Understanding Analysis Solutions

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Contents

Chapter 1. The Real Numbers

1.2. Some Preliminaries

Exercise 1.2.1.

(a) Prove that $\sqrt{3}$ is irrational. Does the same argument work to show that $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution.

(a) Suppose there was a rational number $p = \frac{m}{n}$, which we may assume is in lowest terms, such that $p^2 = 3$, i.e. such that $m^2 = 3n^2$. It follows that m^2 is divisible by 3; we claim that this implies that m is divisible by 3. Indeed, for any $k \in \mathbf{Z}$ we have

$$(3k+1)^2 = 3(3k^2 + 2k) + 1 \quad \text{and} \quad (3k+2)^2 = 3(3k^2 + 4k + 1) + 1.$$

Since m is of the form $3k+1$ or $3k+2$ for some integer k if m is not divisible by 3, it follows that

$$\text{if } m \text{ is not divisible by } 3, \text{then } m^2 \text{ is not divisible by } 3;$$

the contrapositive of this statement proves our claim.

Thus we may write $m = 3k$ for some $k \in \mathbf{Z}$ and substitute this into the equation $m^2 = 3n^2$ to obtain the equation $n^2 = 3k^2$, from which it follows that n is also divisible by 3, contradicting our assumption that m and n had no common factors. We may conclude that there is no rational number whose square is 3.

The same argument works to show that there is no rational number whose square is 6; the crux of this argument is the implication

$$\text{if } m^2 \text{ is divisible by } 6, \text{then } m \text{ is divisible by } 6.$$

This can be seen using what we have already proved. If m^2 is divisible by $6 = 2 \cdot 3$, then m^2 is divisible by 2 and 3. It follows that m is divisible by 2 and 3 and hence that m is divisible by 6.

(b) The argument breaks down when we try to assert that

$$\text{if } m^2 \text{ is divisible by } 4, \text{then } m \text{ is divisible by } 4.$$

This implication is false. For example, $2^2 = 4$ is divisible by 4 but 2 is not divisible by 4.

Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Suppose there was a rational number $r = \frac{m}{n}$, which we may assume is in lowest terms with $n > 0$, such that $2^r = 3$. This implies that $2^m = 3^n$. Since $n > 0$ gives $3^n \geq 3$ and $2^m < 2$ for $m \leq 0$, it must be the case that $m > 0$. It follows that the left-hand side of the equation $2^m = 3^n$ is a positive even integer whereas the right-hand side is a positive odd integer, which is a contradiction. We may conclude that there is no rational number r such that $2^r = 3$.

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

(a) This is false, as Example 1.2.2 shows.

(b) This is true and we can use the following lemma to prove it.

Lemma L.1. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive integers, i.e. $a_{n+1} \leq a_n$ and $a_n \geq 1$ for all $n \in \mathbf{N}$, then $(a_n)_{n=1}^{\infty}$ must be eventually constant. That is, there exists an $N \in \mathbf{N}$ such that $a_n = a_N$ for all $n \geq N$.

Proof. Let $A = \{a_n : n \in \mathbf{N}\}$, which is non-empty and bounded below by 1. It follows from the well-ordering principle that A has a least element, say $\min A = a_N$ for some $N \in \mathbf{N}$. Let $n > N$ be given. It cannot be the case that $a_n < a_N$, since this would contradict that a_N is the least element of A, so we must have $a_n \geq a_N$. By assumption $a_n \leq a_N$ and so we may conclude that $a_n = a_N$. □

Consider the sequence $(|A_n|)_{n=1}^{\infty}$, where $|A_n|$ is the number of elements contained in A_n. Because each A_n is finite and non-empty, this is a sequence of positive integers. Furthermore, this sequence is decreasing since the sets $(A_n)_{n=1}^{\infty}$ are nested:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots.$$

We may now invoke Lemma L.1 to obtain an $N \in \mathbf{N}$ such that $|A_n| = |A_N|$ for all $n \geq N$. Combining this equality with the inclusion $A_n \subseteq A_N$ for each $n \geq N$, we see that $A_n = A_N$ for all $n \geq N$. It follows that $\bigcap_{n=1}^{\infty} A_n = A_N$, which by assumption is finite and non-empty.

(c) This is false: let $A = B = \emptyset$ and $C = \{0\}$ and observe that

$$A \cap (B \cup C) = \emptyset \neq \{0\} = (A \cap B) \cup C.$$

(d) This is true, since

$$x \in A \cap (B \cap C)) \;\Leftrightarrow\; x \in A \text{ and } x \in (B \cap C) \;\Leftrightarrow\; x \in A \text{ and } (x \in B \text{ and } x \in C)$$
$$\Leftrightarrow\; (x \in A \text{ and } x \in B) \text{ and } x \in C \;\Leftrightarrow\; x \in (A \cap B) \text{ and } x \in C \;\Leftrightarrow\; x \in (A \cap B) \cap C,$$

where we have used that logical conjunction ("and") is associative for the third equivalence. It follows that x belongs to $A \cap (B \cap C)$ if and only if x belongs to $(A \cap B) \cap C$, which is to say that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) This is true, since

$$x \in A \cap (B \cup C) \;\Leftrightarrow\; x \in A \text{ and } x \in (B \cup C) \;\Leftrightarrow\; x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\Leftrightarrow\; (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \;\Leftrightarrow\; x \in (A \cap B) \text{ or } x \in (A \cap C)$$
$$\Leftrightarrow\; x \in (A \cap B) \cup (A \cap C),$$

where we have used that logical conjunction ("and") distributes over logical disjunction ("or") for the third equivalence. It follows that x belongs to $A \cap (B \cup C)$ if and only if x belongs to $(A \cap B) \cup (A \cap C)$, which is to say that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution. Arrange $\mathbf{N}$ in a grid like so:

A_1	A_2	A_3	A_4	$\cdots$
1	3	6	10	$\cdots$
2	5	9	14	$\cdots$
4	8	13	19	$\cdots$
7	12	18	25	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now take A_i to be the set of numbers appearing in the i^{th} column.

Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^{\mathrm{c}}$, explain why $x \in A^{\mathrm{c}} \cup B^{\mathrm{c}}$. This shows that $(A \cap B)^{\mathrm{c}} \subseteq A^{\mathrm{c}} \cup B^{\mathrm{c}}$.

(b) Prove the reverse inclusion $(A \cap B)^{\mathrm{c}} \supseteq A^{\mathrm{c}} \cup B^{\mathrm{c}}$, and conclude that $(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}$.

(c) Show $(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}$ by demonstrating inclusion both ways.

Solution.

(a) Observe that

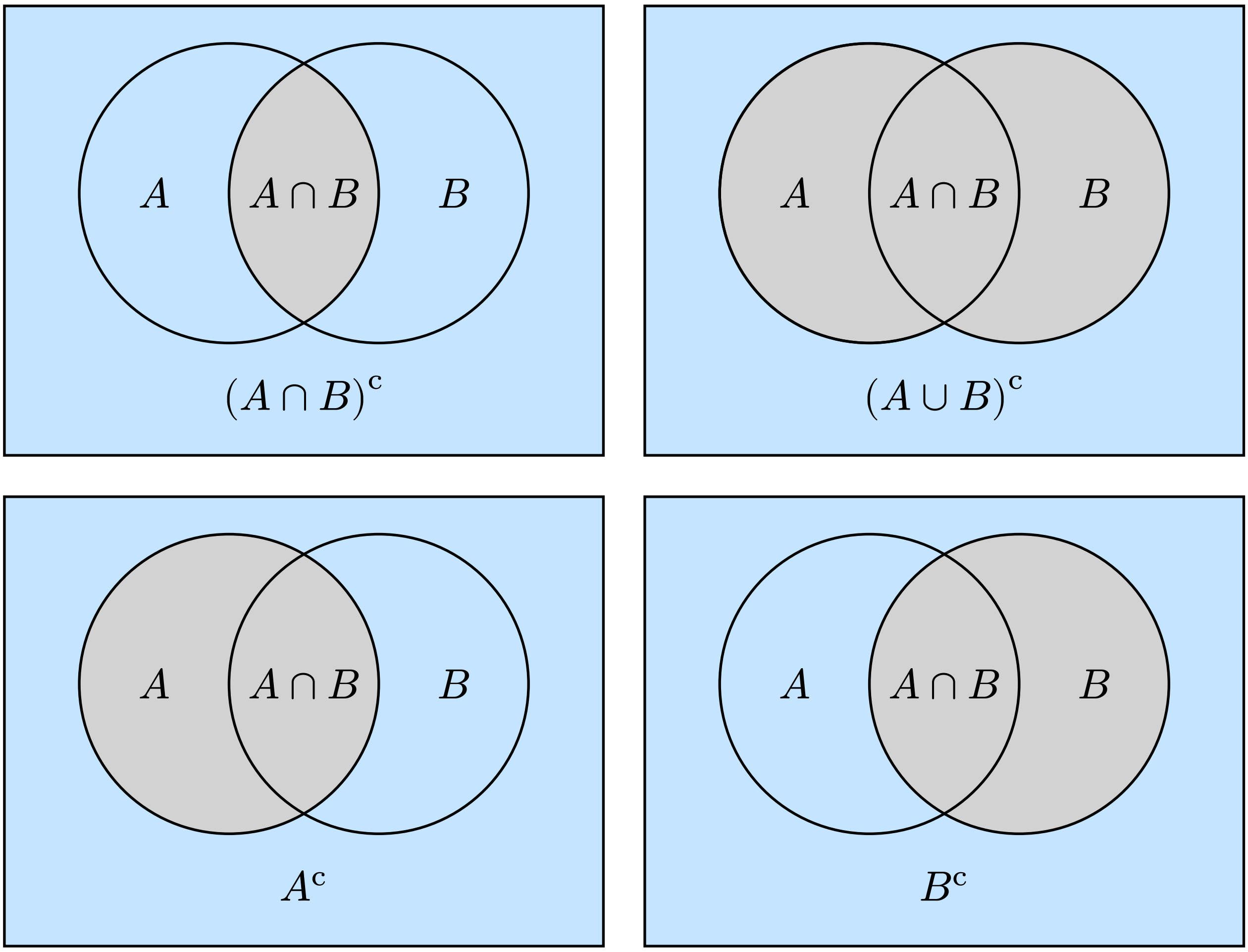
$$x \in (A \cap B)^{\mathrm{c}} \quad \Leftrightarrow \quad x \notin A \cap B \quad \Leftrightarrow \quad \text{not } (x \in A \text{ and } x \in B)$$
$$\Leftrightarrow \quad x \notin A \text{ or } x \notin B \quad \Leftrightarrow \quad x \in A^{\mathrm{c}} \cup B^{\mathrm{c}}$$

(b) See part (a).

(c) The proof is similar to the one given in part (a).

$$x \in (A \cup B)^{\mathrm{c}} \quad \Leftrightarrow \quad x \notin A \cup B \quad \Leftrightarrow \quad \text{not } (x \in A \text{ or } x \in B)$$
$$\Leftrightarrow \quad x \notin A \text{ and } x \notin B \quad \Leftrightarrow \quad x \in A^{\mathrm{c}} \cap B^{\mathrm{c}}$$

The following Venn diagrams help to visualize De Morgan's Laws. The blue regions are included and the grey regions are excluded.



Exercise 1.2.6.

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution.

(a) First suppose that a and b are both non-negative, so that $a+b$ is also non-negative; it follows that $|a+b| = a+b$ and $|a|+|b| = a+b$. Thus the triangle inequality in this case reduces to the evidently true statement $a+b \leq a+b$.

Now suppose that a and b are both negative, so that $a+b$ is also negative; it follows that $|a+b| = -a-b$ and $|a|+|b| = -a-b$. Thus the triangle inequality in this case reduces to the evidently true statement $-a-b \leq -a-b$.

(b) Starting from the true statement $ab \leq |ab|$ and using that $a^2 = |a|^2$ and $|ab| = |a||b|$ for any real numbers a and b, observe that

$$2ab \leq 2|ab| \quad \Leftrightarrow \quad a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2$$
$$\Leftrightarrow \quad (a+b)^2 \leq (|a|+|b|)^2 \quad \Leftrightarrow \quad |a+b|^2 \leq (|a|+|b|)^2.$$

Because both $|a+b|$ and $|a|+|b|$ are non-negative, the inequality $|a+b|^2 \leq (|a|+|b|)^2$ is equivalent to $|a+b| \leq |a|+|b|$, as desired.

(c) We apply the triangle inequality twice:

$$|a-b| = |a-c+c-b| \leq |a-c| + |c-b| \leq |a-c| + |c-d| + |d-b|.$$

(d) Using the triangle inequality and the fact that $|-a| = |a|$ for any $a \in \mathbf{R}$, we find that

$$|a| = |a-b+b| \leq |a-b| + |b| \quad \Leftrightarrow \quad |a| - |b| \leq |a-b|,$$
$$|b| = |b-a+a| \leq |b-a| + |a| = |a-b| + |a| \quad \Leftrightarrow \quad |b| - |a| \leq |a-b|.$$

Because $||a|-|b||$ equals either $|a|-|b|$ or $|b|-|a|$, it follows that $||a|-|b|| \leq |a-b|$.

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution.

(a) Some straightforward calculations reveal that

$$f(A) = [0, 4], \qquad f(A \cap B) = [1, 4], \qquad f(A \cup B) = [0, 16],$$

$$f(B) = [1, 16], \qquad f(A) \cap f(B) = [1, 4], \qquad f(A) \cup f(B) = [0, 16].$$

From this we see that $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$.

(b) Let $A = \{-1\}$ and $B = \{1\}$ and note that $f(A \cap B) = f(\emptyset) = \emptyset$, but

$$f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\} \neq \emptyset.$$

(c) Observe that

$$y \in g(A \cap B) \quad \Leftrightarrow \quad y = g(x) \text{ for some } x \in A \cap B$$

$$\Rightarrow \quad (y = g(x_1) \text{ for some } x_1 \in A) \text{ and } (y = g(x_2) \text{ for some } x_2 \in B)$$

$$\Leftrightarrow \quad y \in g(A) \text{ and } y \in g(B) \quad \Leftrightarrow \quad y \in g(A) \cap g(B).$$

It follows that y belongs to $g(A) \cap g(B)$ whenever y belongs to $g(A \cap B)$, which is to say that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) We always have $g(A \cup B) = g(A) \cup g(B)$; indeed,

$$y \in g(A \cup B) \quad \Leftrightarrow \quad y = g(x) \text{ for some } x \in A \cup B$$

$$\Leftrightarrow \quad y = g(x) \text{ for some } x \text{ such that } (x \in A \text{ or } x \in B)$$

$$\Leftrightarrow \quad (y = g(x_1) \text{ for some } x_1 \in A) \text{ or } (y = g(x_2) \text{ for some } x_2 \in B)$$

$$\Leftrightarrow \quad y \in g(A) \text{ or } y \in g(B) \quad \Leftrightarrow \quad y \in g(A) \cup g(B).$$

It follows that $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

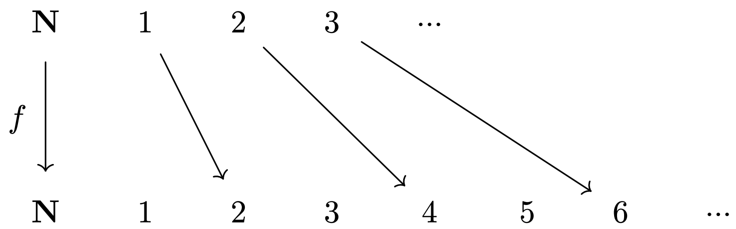
(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

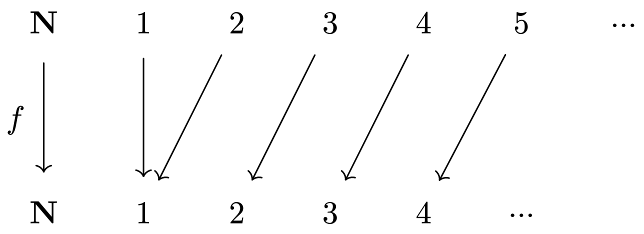
(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

Solution.

(a) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(n) = 2n$. Notice that f is injective since $n = m$ if and only if $2n = 2m$, but f is not surjective since the range of f contains only even numbers.

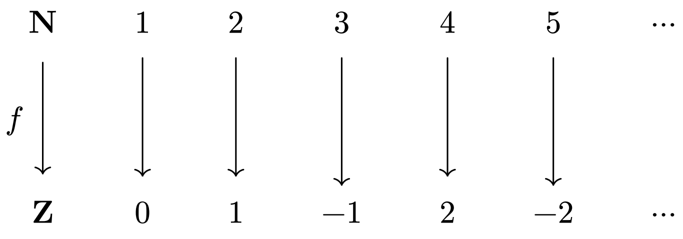


(b) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(1) = 1$ and $f(n) = n - 1$ for $n \geq 2$. Notice that $f(n+1) = n$ for any $n \in \mathbf{N}$, so that f is surjective, but f is not injective since $f(1) = f(2) = 1$.



(c) Let $f : \mathbf{N} \to \mathbf{Z}$ be given by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$



To see that f is injective, let $n \neq m$ be given and consider these cases.

Case 1. If n and m are both even, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $\frac{n}{2} \neq \frac{m}{2}$.

Case 2. If n and m are both odd, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $-\frac{n-1}{2} \neq -\frac{m-1}{2}$.

Case 3. If n and m have opposite signs, say n is even and m is odd, then $f(n) \neq f(m)$ since $f(n) > 0$ and $f(m) \leq 0$.

To see that f is surjective, let $n \in \mathbf{Z}$ be given. If $n > 0$ then $f(2n) = n$, and if $n \leq 0$ then $f(-2n+1) = n$.

Exercise 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution.

(a) Some straightforward calculations reveal that

$$f^{-1}(A) = [-2, 2], \qquad f^{-1}(A \cap B) = [-1, 1], \qquad f^{-1}(A \cup B) = [-2, 2],$$

$$f^{-1}(B) = [-1, 1], \qquad f^{-1}(A) \cap f^{-1}(B) = [-1, 1], \qquad f^{-1}(A) \cup f^{-1}(B) = [-2, 2].$$

From this we see that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \quad \text{and} \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

(b) Observe that

$$\begin{aligned} x \in g^{-1}(A \cap B) \quad &\Leftrightarrow \quad g(x) \in A \cap B \quad \Leftrightarrow \quad (g(x) \in A) \text{ and } (g(x) \in B) \\ &\Leftrightarrow \quad (x \in g^{-1}(A)) \text{ and } (x \in g^{-1}(B)) \quad \Leftrightarrow \quad x \in g^{-1}(A) \cap g^{-1}(B). \end{aligned}$$

Similarly,

$$\begin{aligned} x \in g^{-1}(A \cup B) \quad &\Leftrightarrow \quad g(x) \in A \cup B \quad \Leftrightarrow \quad (g(x) \in A) \text{ or } (g(x) \in B) \\ &\Leftrightarrow \quad (x \in g^{-1}(A)) \text{ or } (x \in g^{-1}(B)) \quad \Leftrightarrow \quad x \in g^{-1}(A) \cup g^{-1}(B). \end{aligned}$$

Exercise 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \varepsilon$ for every $\varepsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

Solution.

(a) This is false; the implication

$$\text{if } a < b + \varepsilon \text{ for every } \varepsilon > 0, \text{then } a < b$$

does not hold. The problem occurs when we consider the case where $a = b$. For example, we certainly have $1 < 1 + \varepsilon$ for every $\varepsilon > 0$ but of course $1 < 1$ is false.

(b) See part (a).

(c) This is true. The implication

$$\text{if } a \leq b, \text{then } a < b + \varepsilon \text{ for every } \varepsilon > 0$$

follows since $a \leq b < b + \varepsilon$ for every $\varepsilon > 0$ and the implication

$$\text{if } a > b, \text{then } a \geq b + \varepsilon \text{ for some } \varepsilon > 0$$

can be seen by taking $\varepsilon = a - b > 0$, so that $b + \varepsilon = a \leq a$.

Exercise 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution.

(a) The negated statement is:

$$\text{there exist real numbers } a < b \text{ such that } a + \frac{1}{n} \geq b \text{ for all } n \in \mathbf{N}.$$

The original statement is true and follows from the Archimedean Property (Theorem 1.4.2).

(b) The negated statement is:

$$\text{for all } x > 0, \text{there exists an } n \in \mathbf{N} \text{ such that } \frac{1}{n} \leq x.$$

The negated statement is true and again follows from the Archimedean Property (Theorem 1.4.2).

(c) The negated statement is:

there are two distinct real numbers with no rational number between them.

The original statement is true; this is the density of **Q** in **R** (Theorem 1.4.3).

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show that the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_n > -6$. Since $y_1 = 6$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} = \tfrac{2}{3}y_n - 2 > \tfrac{2}{3}(-6) - 2 = -6,$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_{n+1} \leq y_n$. Since $y_1 = 6$ and $y_2 = 2$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+2} = \tfrac{2}{3}y_{n+1} - 2 \leq \tfrac{2}{3}y_n - 2 = y_{n+1},$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^{\mathrm{c}} = A_1^{\mathrm{c}} \cap A_2^{\mathrm{c}} \cap \cdots \cap A_n^{\mathrm{c}}$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathrm{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathrm{c}},$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $(A_1 \cup \cdots \cup A_n)^{\mathrm{c}} = A_1^{\mathrm{c}} \cap \cdots \cap A_n^{\mathrm{c}}$ for any sets $A_1, \ldots, A_n$. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$, let $A_1, \ldots, A_n, A_{n+1}$ be given, and observe that

$$\begin{aligned}
(A_1 \cup \cdots \cup A_n \cup A_{n+1})^{\mathrm{c}} &= ((A_1 \cup \cdots \cup A_n) \cup (A_{n+1}))^{\mathrm{c}} \\
&= (A_1 \cup \cdots \cup A_n)^{\mathrm{c}} \cap A_{n+1}^{\mathrm{c}} && \text{(Exercise 1.2.5)} \\
&= A_1^{\mathrm{c}} \cap \cdots \cap A_n^{\mathrm{c}} \cap A_{n+1}^{\mathrm{c}}, && \text{(induction hypothesis)}
\end{aligned}$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) Let $B_i = \{i, i+1, i+2, \ldots\}$, so that

$$B_1 = \{1, 2, 3, \ldots\}, \quad B_2 = \{2, 3, 4, \ldots\}, \quad B_3 = \{3, 4, 5, \ldots\}, \quad \text{etc.}$$

It is straightforward to verify that $\bigcap_{i=1}^{n} B_i = B_n \neq \emptyset$ for any $n \in \mathbf{N}$. However, as Example 1.2.2 shows, the intersection $\bigcap_{i=1}^{\infty} B_i$ is empty.

(c) Observe that

$$x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathrm{c}} \quad \Leftrightarrow \quad x \notin \bigcup_{i=1}^{\infty} A_i \quad \Leftrightarrow \quad x \notin A_i \text{ for every } i \in \mathbf{N} \quad \Leftrightarrow \quad x \in \bigcap_{i=1}^{\infty} A_i^{\mathrm{c}}.$$

It follows that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathrm{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathrm{c}}.$$

1.3. The Axiom of Completeness

Exercise 1.3.1.

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution.

(a) A real number t is the *greatest lower bound* for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

  (i) t is a lower bound for A;

  (ii) if b is any lower bound for A, then $b \leq t$.

(b) Here is a version of Lemma 1.3.8 for greatest lower bounds.

Lemma L.2. If $t \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$, then $t = \inf A$ if and only if for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $a < t + \varepsilon$.

Proof. First, let us prove the implication

if $t = \inf A$, then for every $\varepsilon > 0$ there exists an $a \in A$ such that $a < t + \varepsilon$

by proving the contrapositive statement

if there exists an $\varepsilon > 0$ such that $t + \varepsilon \leq a$ for every $a \in A$, then $t \neq \inf A$.

If such an $\varepsilon > 0$ exists, then $t + \varepsilon$ is a lower bound for A strictly greater than t; it follows that t is not the greatest lower bound for A, i.e. $t \neq \inf A$.

Now let us prove the converse:

if for every $\varepsilon > 0$ there exists an $a \in A$ such that $a < t + \varepsilon$, then $t = \inf A$.

Suppose $b \in \mathbf{R}$ is such that $b > t$. Letting $\varepsilon = b - t > 0$, we are guaranteed the existence of an $a \in A$ such that $a < t + \varepsilon = b$; it follows that b is not a lower bound for A. This proves the contrapositive of criterion (ii) in part (a) and we may conclude that $t = \inf A$. □

Exercise 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution.

(a) Let $B = \{0\}$ and notice that $\inf B = \sup B = 0$.

(b) This is impossible. To see this, let us first use induction to show that any non-empty finite subset of $\mathbf{R}$ contains a minimum and a maximum element.

Lemma L.3. If $E \subseteq \mathbf{R}$ is non-empty and finite, then E contains a minimum and a maximum element.

Proof. For $n \in \mathbf{N}$, let $P(n)$ be the statement that any subset of $\mathbf{R}$ containing n elements has a minimum and a maximum element. For the base case $P(1)$, simply observe that $\min\{x\} = \max\{x\} = x$ for any $x \in \mathbf{R}$.

Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and let $E \subseteq \mathbf{R}$ be a set containing $n+1$ elements. Fix some $x \in E$ and consider the set $F = E \setminus \{x\}$, which contains n elements. Our induction hypothesis guarantees the existence of a minimum element $a = \min F$ and a maximum element $b = \max F$, which must satisfy $a \leq b$. There are now three cases; the conclusion is each case is straightforward to verify.

Case 1. If $x < a$, then $\min E = x$ and $\max E = b$.

Case 2. If $x > b$, then $\min E = a$ and $\max E = x$.

Case 3. If $a \leq x \leq b$, then $\min E = a$ and $\max E = b$.

In any case, the set E has a minimum and a maximum element, i.e. $P(n+1)$ holds. This completes the induction step and the proof. □

It is immediate from the definition of the supremum and the maximum of a set $E \subseteq \mathbf{R}$ that if $\max E$ exists then $\sup E = \max E$ (see Exercise 1.3.7); similarly, if $\min E$ exists then $\inf E = \min E$. It follows that the given request is impossible: if $E \subseteq \mathbf{R}$ is finite, then Lemma L.3 implies that $\min E = \inf E$ and $\max E = \sup E$ both exist and hence E contains both its infimum and its supremum.

(c) Consider the bounded set $E = \{p \in \mathbf{Q} : 0 < p \leq 1\}$, which satisfies $\sup E = 1 \in E$ and $\inf E = 0 \notin E$.

Exercise 1.3.3.

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution.

(a) B is non-empty since A is bounded below, and B is bounded above by any $x \in A$; there exists at least one such x since A is non-empty. It follows from the Axiom of Completeness that $\sup B$ exists. To see that $\sup B = \inf A$, we need to show that $\sup B$ satisfies criteria (i) and (ii) from Exercise 1.3.1 (a).

   (i) First we need to prove that $\sup B$ is a lower bound of A, i.e. if $x \in A$ then $\sup B \leq x$. We will prove the contrapositive statement: if $x < \sup B$ then $x \notin A$. If x is strictly less than $\sup B$, then x cannot be an upper bound of B. Thus there exists some $b \in B$ such that $x < b$. Since b is a lower bound of A, it follows that $x \notin A$.

   (ii) Now we need to show that $\sup B$ is the greatest lower bound of A. Indeed, suppose $y \in \mathbf{R}$ is a lower bound of A, so that $y \in B$; it follows that $y \leq \sup B$.

   We may conclude that $\sup B = \inf A$.

(b) Part (a) shows that the existence of the greatest lower bound for non-empty bounded below subsets of $\mathbf{R}$ is implied by the Axiom of Completeness; adding this existence as part of the Axiom of Completeness would be redundant.

Exercise 1.3.4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution.

(a) Let $n \in \mathbf{N}$ be given. For each $k \in \{1, \ldots, n\}$, the Axiom of Completeness guarantees that $\sup A_k$ exists. By Lemma L.3, the finite set $\{\sup A_1, \ldots, \sup A_k\}$ has a maximum element, say M; we claim that $\sup\left(\bigcup_{k=1}^{n} A_k\right) = M$. To prove this, we must verify criteria (i) and (ii) from Definition 1.3.2.

   (i) If $x \in \bigcup_{k=1}^{n} A_k$, then $x \in A_k$ for some $k \in \{1, \ldots, n\}$; it follows that $x \leq \sup A_k \leq M$. Since x was arbitrary, we see that M is an upper bound for $\bigcup_{k=1}^{n} A_k$.

(ii) If $b \in \mathbf{R}$ is an upper bound for $\bigcup_{k=1}^{n} A_k$, then b must be an upper bound for each A_k. It follows that $\sup A_k \le b$ for each $k \in \{1, ..., n\}$ and thus $M \le b$.

We may conclude that $\sup\left(\bigcup_{k=1}^{n} A_k\right) = M$.

(b) The proof given above does not extend to the infinite case, since the set $\{\sup A_1, \sup A_2, ...\}$ need not have a maximum. Indeed, it may be the case that $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$ does not exist. For example, let $A_k = \{k\}$, which is non-empty and bounded above with $\sup A_k = k$, but $\bigcup_{k=1}^{\infty} A_k = \mathbf{N}$, which does not have a supremum in $\mathbf{R}$.

Exercise 1.3.5. As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \ge 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution.

(a) If $c = 0$ then the result is clear, so suppose that $c > 0$. For any $x \in A$, notice that

$$x \le \sup A \quad \Leftrightarrow \quad cx \le c \sup A.$$

This demonstrates that $c \sup A$ is an upper bound of cA.

Now observe that

$$b \in \mathbf{R} \text{ is an upper bound of } cA \quad \Leftrightarrow \quad cx \le b \text{ for all } x \in A$$
$$\Leftrightarrow \quad x \le c^{-1}b \text{ for all } x \in A \quad \Leftrightarrow \quad c^{-1}b \text{ is an upper bound of } A.$$

It follows that $\sup A \le c^{-1}b$ and hence that $c \sup A \le b$. We may conclude that $\sup(cA) = c \sup A$.

(b) If $c < 0$ and $\inf A$ exists then $\sup(cA) = c \inf A$. The proof is similar to part (a). For any $x \in A$, we have

$$\inf A \le x \quad \Leftrightarrow \quad cx \le c \inf A,$$

so that $c \inf A$ is an upper bound of cA.

Observe that

$$b \in \mathbf{R} \text{ is an upper bound of } cA \quad \Leftrightarrow \quad cx \le b \text{ for all } x \in A$$
$$\Leftrightarrow \quad c^{-1}b \le x \text{ for all } x \in A \quad \Leftrightarrow \quad c^{-1}b \text{ is an lower bound of } A.$$

It follows that $c^{-1}b \le \inf A$ and hence that $c \inf A \le b$. We may conclude that $\sup(cA) = c \inf A$.

If $\inf A$ doesn't exist then $\sup(cA)$ doesn't exist either, since for $c < 0$ the set A is bounded below if and only if cA is bounded above. For example, $A = (-\infty, 0)$ and $c = -1$ gives $cA = (0, \infty)$.

Exercise 1.3.6. Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution.

(a) For any $a \in A$ and $b \in B$ we have $a \leq s$ and $b \leq t$. It follows that $a + b \leq s + t$ and thus $s + t$ is an upper bound of $A + B$.

(b) For any $b \in B$ we have $a + b \leq u$, which gives $b \leq u - a$. This demonstrates that $u - a$ is an upper bound for B and so it follows that $t \leq u - a$.

(c) Part (b) implies that for any $a \in A$ we have $t \leq u - a$, which gives $a \leq u - t$. This shows that $u - t$ is an upper bound of A and it follows that $s \leq u - t$, i.e. $s + t \leq u$. Since u was an arbitrary upper bound of $A + B$, we may conclude that

$$\sup(A + B) = s + t = \sup A + \sup B.$$

(d) Let $\varepsilon > 0$ be given. By Lemma 1.3.8, there exist elements $a \in A$ and $b \in B$ such that $s - \frac{\varepsilon}{2} < a$ and $t - \frac{\varepsilon}{2} < b$, which implies that $s + t - \varepsilon < a + b$. We showed in part (a) that $s + t$ is an upper bound of $A + B$, so we may invoke Lemma 1.3.8 to conclude that $\sup(A + B) = \sup A + \sup B$.

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution. Let $b \in \mathbf{R}$ be an upper bound of A. Since $a \in A$, we must have $a \leq b$; it follows that $a = \sup A$.

Exercise 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbf{N}\}$.

(d) $\{m/(m + n) : m, n \in \mathbf{N}\}$.

Solution.

(a) The supremum is 1 and the infimum is 0.

(b) The supremum is 1 and the infimum is -1.

(c) The supremum is $\frac{1}{3}$ and the infimum is $\frac{1}{4}$.

(d) The supremum is 1 and the infimum is 0.

Exercise 1.3.9.

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution.

(a) Let $\varepsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists some $b \in B$ such that $\sup B - \varepsilon = \sup A < b$. It follows that b is an upper bound of A.

(b) If we let $A = B = (0, 1)$ then $\sup A = \sup B = 1$, but no element of B is an upper bound of A.

Exercise 1.3.10 (Cut Property). The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution.

(a) Suppose that A and B are non-empty disjoint subsets of $\mathbf{R}$ such that $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$. Notice that A is non-empty (by assumption) and bounded above (because B is non-empty); the Axiom of Completeness then implies that $c = \sup A$ exists. It follows that $x \leq c$ for all $x \in A$ and, since each element of B is an upper bound of A, we also have $x \geq c$ for all $x \in B$.

(b) Suppose that $E \subseteq \mathbf{R}$ is non-empty and bounded above. Define

$$A = \{a \in \mathbf{R} : a \text{ is not an upper bound of } E\}$$
$$\text{and} \quad B = A^{c} = \{b \in \mathbf{R} : b \text{ is an upper bound of } E\}.$$

Notice that B is non-empty as E is bounded above and A is non-empty because $x - 1 \in A$ for any $x \in E$; we are guaranteed the existence of at least one $x \in E$ as E is non-empty. Furthermore, A and B are evidently disjoint and satisfy $A \cup B = \mathbf{R}$.

Let $a \in A$ and $b \in B$ be given. Since a is not an upper bound of E there exists some $x \in E$ such that $a < x$ and since b is an upper bound of E, we must then have $x \leq b$; it follows that $a < b$. We may now invoke the Cut Property to obtain a $c \in \mathbf{R}$ such that $x \leq c$ for all $x \in A$ and $x \geq c$ for all $x \in B$.

We claim that $c = \sup E$. Since $A \cup B = \mathbf{R}$ and $A \cap B = \emptyset$, exactly one of $c \in A$ or $c \in B$ holds. Suppose that $c \in A$, i.e. c is not an upper bound of E, which is the case if and only if there is some $t \in E$ such that $c < t$. Observe that $y = \frac{c+t}{2}$ satisfies $c < y < t$, so that $y \in A$—but this contradicts the fact that $x \leq c$ for all $x \in A$.

So it must be the case that $c \in B$, i.e. c is an upper bound of E. The Cut Property guarantees that $c \leq x$ for all $x \in B$. In other words, c is less than all other upper bounds of E; we may conclude that $c = \sup E$.

(c) A concrete example is given in the following lemma.

Lemma L.4. The sets

$$A = \{p \in \mathbf{Q} : p < 0 \text{ or } p^2 < 2\} \quad \text{and} \quad B = \{p \in \mathbf{Q} : p > 0 \text{ and } p^2 > 2\}$$

satisfy the following properties:

(i) A and B are non-empty, $A \cup B = \mathbf{Q}$, and $A \cap B = \emptyset$;

(ii) $p < q$ for all $p \in A$ and $q \in B$;

(iii) A has no maximum element and B has no minimum element.

Proof.

(i) Certainly A and B are non-empty. The negation of the statement "$p < 0$ or $p^2 < 2$" is "$p > 0$ and $p^2 \geq 2$"; by Theorem 1.1.1, this negated statement is equivalent to "$p > 0$ and $p^2 > 2$" for $p \in \mathbf{Q}$. Thus $B = \mathbf{Q} \setminus A$, from which it follows that $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$.

(ii) Let $p \in A$ and $q \in B$ be given. If $p \leq 0$ then certainly $p < q$, so suppose that $p > 0$. It must then be the case that $p^2 < 2$, whence $p^2 < q^2$. Since p and q are positive, this implies that $p < q$.

(iii) Let $p \in A$ be given. We need to show that there exists some $q \in A$ such that $p < q$. If $p \leq 0$, we can take $q = 1$; if $p > 0$, so that $p^2 < 2$, then define

$$q = p + \frac{2-p^2}{p+2} = \frac{2p+2}{p+2}. \tag{1}$$

Notice that $0 < \frac{2-p^2}{p+2}$, since $p^2 < 2$, from which it follows that $p < q$. A straightforward calculation yields

$$2 - q^2 = \frac{2(2-p^2)}{(p+2)^2};$$

again using that $p^2 < 2$, we see that $2 - q^2 > 0$ and thus $q \in A$.

Now let $p \in B$ be given. We need to show that there exists some $q \in B$ such that $q < p$. In fact, we can define q by equation (1) again; an argument similar to the one just given shows that $q < p$ and $q \in B$. □

Parts (i) and (ii) of Lemma L.4 show that the sets A and B satisfy the hypotheses of the Cut Property. If the Cut Property held for $\mathbf{Q}$, then we would be able to obtain a $c \in \mathbf{Q}$ such that $p \leq c$ for all $p \in A$ and $c \leq q$ for all $q \in B$. Since $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$, this implies that c is either the maximum of A or the minimum of B—but this contradicts part (iii) of Lemma L.4. We may conclude that the Cut Property does not hold for $\mathbf{Q}$.

Exercise 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution.

(a) This is true. The Axiom of Completeness guarantees that $\sup A$ and $\sup B$ both exist. Furthermore, since each element of A is an element of B, any upper bound of B must be an upper bound of A also. In particular, $\sup B$ must be an upper bound of A; it follows that $\sup A \leq \sup B$.

(b) This is true. Let $c = \frac{\sup A + \inf B}{2}$, so that $\sup A < c < \inf B$, and notice that for any $a \in A$ and $b \in B$ we have

$$a \leq \sup A < c < \inf B \leq b.$$

(c) This is false. Consider $A = (-1, 0)$ and $B = (0, 1)$, and notice that $c = 0$ satisfies $a < c < b$ for all $a \in A$ and $b \in B$, but $\sup A = \inf B = 0$.

1.4. Consequences of Completeness

Exercise 1.4.1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.

(b) Show tha if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a+t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Solution.

(a) Suppose $a = \frac{m}{n}$ and $b = \frac{p}{q}$ and observe that

$$ab = \frac{mp}{nq} \quad \text{and} \quad a+b = \frac{mq+np}{nq},$$

which are rational numbers.

(b) Let $a \in \mathbf{Q}$ be fixed. We want to prove that

$$t \in \mathbf{I} \quad \Rightarrow \quad a+t \in \mathbf{I}.$$

To do this, we will prove the contrapositive statement

$$a+t \in \mathbf{Q} \quad \Rightarrow \quad t \in \mathbf{Q}.$$

Simply observe that $t = (a+t) - a$; it follows from part (a) that $t \in \mathbf{Q}$.

Similarly, let $a \in \mathbf{Q}$ be non-zero. We can show that

$$at \in \mathbf{Q} \quad \Rightarrow \quad t \in \mathbf{Q}$$

by observing that $t = a^{-1}(at)$ and appealing to part (a) to conclude that $t \in \mathbf{Q}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. For example, $-\sqrt{2}$ and $\sqrt{2}$ are irrational numbers, but their sum is the rational number 0 and their product is the rational number -2. The sum or product of two irrational numbers may be irrational. For example, it can be shown that $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}\sqrt{3} = \sqrt{6}$ are irrational:

- For the irrationality of $\sqrt{6}$, see Exercise 1.2.1 (a).
- For the irrationality of $\sqrt{2}+\sqrt{3}$, observe that $\sqrt{2}+\sqrt{3}$ is a root of the polynomial $x^4 - 10x^2 + 1$. The rational root theorem says that the only possible rational roots of this polynomial are ± 1—but neither of these solve the equation $x^4 - 10x^2 + 1 = 0$.

So in general, we cannot say anything about the sum or product of two irrational numbers without more information.

Exercise 1.4.2. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution. If s is not an upper of A then there must exist some $x \in A$ such that $s < x$. By the Archimedean Property (Theorem 1.4.2), there then exists a natural number n such that $s + \frac{1}{n} < x$, which implies that $s + \frac{1}{n}$ is not an upper bound of A. Given our hypothesis that $s + \frac{1}{n}$ is an upper bound of A for all $n \in \mathbf{N}$, we see that s must be an upper bound of A.

Now let $\varepsilon > 0$ be given and usng the Archimedean Property (Theorem 1.4.2), pick a natural number n such that $\frac{1}{n} < \varepsilon$. By assumption $s - \frac{1}{n}$ is not an upper bound of A, so there must exist some $x \in A$ such that $s - \frac{1}{n} < x$, which implies that $s - \varepsilon < x$ since $\frac{1}{n} < \varepsilon$. Because $\varepsilon > 0$ was arbitrary, we may invoke Lemma 1.3.8 to conclude that $s = \sup A$.

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closeed for the conclusion of the theorem to hold.

Solution. Certainly $x \notin \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$ if $x \leq 0$, so suppose that $x > 0$. Use the Archimedean Property (Theorem 1.4.2) to choose an $N \in \mathbf{N}$ such that $\frac{1}{N} < x$; it follows that $x \notin \left(0, \frac{1}{N}\right)$ and hence that $x \notin \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. We may conclude that $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right) = \emptyset$.

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. Certainly b is an upper bound of T. Let $\varepsilon > 0$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying

$$\max\{a, b - \varepsilon\} < p < b.$$

It follows that $p \in T$ and $b - \varepsilon < p$ and hence, by Lemma 1.3.8, we may conclude that $\sup T = b$.

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying $a - \sqrt{2} < p < b - \sqrt{2}$, which gives $a < p + \sqrt{2} < b$. Since $p + \sqrt{2}$ is irrational (Exercise 1.4.1 (b)), the corollary is proved.

Exercise 1.4.6. Recall that a set B is *dense* in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution.

(a) This set is not dense in $\mathbf{R}$. For $1 \leq q \leq 10$, observe that if $p \geq 1$ then $\frac{p}{q} \geq \frac{1}{10}$, if $p \leq -1$ then $\frac{p}{q} \leq -\frac{1}{10}$, and if $p = 0$ then $\frac{p}{q} = 0$. So there is no element of this set between the real numbers $\frac{1}{1000}$ and $\frac{1}{100}$, for example.

(b) This set is dense in $\mathbf{R}$. Let $a < b$ be given real numbers. Using the Archimedean Property (Theorem 1.4.2), let $n \in \mathbf{N}$ be such that $\frac{1}{n} < b - a$, which implies that $\frac{1}{2^n} < b - a$. Now let p be the smallest integer greater than $2^n a$, so that $p - 1 \leq 2^n a < p$, and observe that

$$2^n a < p \leq 1 + 2^n a < 2^n b;$$

it follows that $\frac{p}{2^n}$ lies between a and b.

(c) This set is not dense in $\mathbf{R}$. If $p > 0$ then

$$10|p| \geq q \quad \Leftrightarrow \quad 10p \geq q \quad \Leftrightarrow \quad \frac{p}{q} \geq \frac{1}{10},$$

and if $p < 0$ then

$$10|p| \geq q \quad \Leftrightarrow \quad -10p \geq q \quad \Leftrightarrow \quad \frac{p}{q} \leq -\frac{1}{10}.$$

We cannot have $p = 0$ since q is a positive integer. Thus there is no element of this set between the real numbers 0 and $\frac{1}{100}$, for example.

Exercise 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution. Assuming that $\alpha^2 - 2 > 0$, the Archimedean Property (Theorem 1.4.2) implies that there is an $n \in \mathbf{N}$ such that

$$\frac{2\alpha}{n} < a^2 - 2 \quad \Leftrightarrow \quad 2 < \alpha^2 - \frac{2\alpha}{n}.$$

Let $\beta = \alpha - \frac{1}{n}$ and note that since $1 \in T$ we have $\alpha \geq 1$ and hence $\beta \geq 0$; it follows that $t \leq \beta$ for all $t \in T$ such that $t < 0$. Now observe that

$$\beta^2 = \left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} > 2,$$

so that for any $t \in T$ we have $t^2 < 2 < \beta^2$. If $t \in T$ is such that $t \geq 0$ then the inequality $t^2 < \beta^2$ implies that $t < \beta$, as β is also non-negative.

We have now shown that $t \leq \beta$ for all $t \in T$, i.e. β is an upper bound for T—but this contradicts the fact that α is the supremum of T since $\beta < \alpha$.

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbf{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution.

(a) Let

$$A = \left\{-\frac{1}{2n} : n \in \mathbf{N}\right\} = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots\right\}$$

$$\text{and} \quad B = \left\{-\frac{1}{2n-1} : n \in \mathbf{N}\right\} = \left\{-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\right\}.$$

Notice that $A \cap B = \emptyset$ and $\sup A = \sup B = 0$, which belongs to neither A nor B.

(b) If we let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbf{N}$, then $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

(c) For $n \in \mathbf{N}$, let $L_n = [n, \infty)$.

(d) This is impossible. To see this, let $(I_n)_{n=1}^{\infty}$ be a sequence of closed bounded intervals satisfying $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for every $N \in \mathbf{N}$. Define $J_N = \bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ and note that any finite intersection of closed bounded intervals is a (possibly empty) closed bounded interval. Thus:

- each J_N is a closed bounded interval;
- these intervals are non-empty and nested, i.e. $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$;
- $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$.

It then follows from the Nested Interval Property (Theorem 1.4.1) that $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$ is non-empty.

1.5. Cardinality

Exercise 1.5.1. Finish the following proof for Theorem 1.5.7.

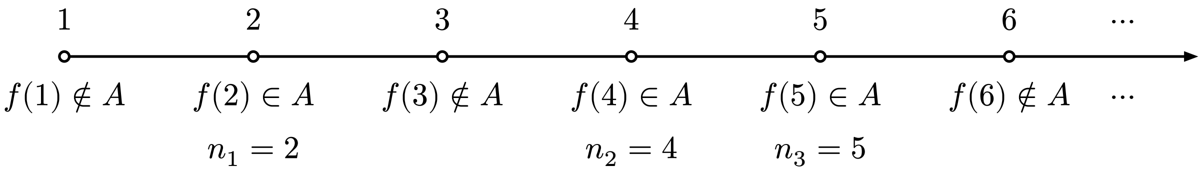
Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$ which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to product a 1-1 function g from $\mathbf{N}$ onto A.

Solution. Given $n_1 = \min f^{-1}(A) = \min\{n \in \mathbf{N} : f(n) \in A\}$, we can construct a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers recursively by defining

$$n_k = \min(f^{-1}(A) \setminus \{n_1, ..., n_{k-1}\}) = \min(\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, ..., n_{k-1}\})$$

for $k \geq 2$. Because A is infinite and f is surjective, the set $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, ..., n_{k-1}\}$ is non-empty (indeed, it must be infinite) for each $k \geq 2$; it follows that each n_k is well-defined. Here is an example construction of the sequence $(n_k)_{k=1}^{\infty}$ for some bijection $f : \mathbf{N} \to B$.



It is clear from this construction that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence.

Define $g : \mathbf{N} \to A$ by $g(k) = f(n_k)$; we claim that g is a bijection. For injectivity, observe that

$$g(\ell) = g(k) \quad \Leftrightarrow \quad f(n_\ell) = f(n_k) \quad \Leftrightarrow \quad n_\ell = n_k \quad \Leftrightarrow \quad \ell = k,$$

where we have used the injectivity of f for the second equivalence and the strict monotonicity of the sequence $(n_k)_{k=1}^{\infty}$ for the third equivalence.

For the surjectivity of g, let $a \in A$ be given. Since f is surjective, there is a positive integer N such that $f(N) = a$; we need to find some $k \in \mathbf{N}$ such that $n_k = N$. It cannot be the case that $N < n_1$, otherwise n_1 would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\}$, so we must have $n_1 \leq N$. Given this, and the fact that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers, there must exist a $k \in \mathbf{N}$ such that $n_k \leq N < n_{k+1}$. In fact, it must be the case that $n_k = N$, otherwise n_{k+1} would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, ..., n_k\}$. Thus $g(k) = f(n_k) = f(N) = a$.

Exercise 1.5.2. Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, ...\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

Solution. The construction does not imply that $\bigcap_{n=1}^{\infty} I_n = \emptyset$; it only guarantees that this intersection does not contain any rational numbers.

Exercise 1.5.3. Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction cannot be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

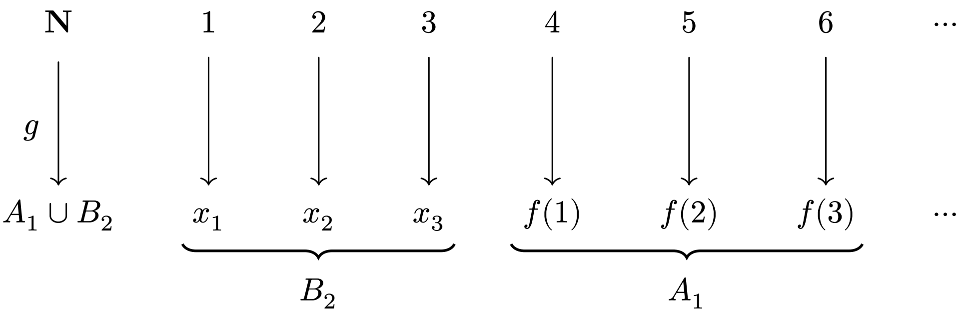
Solution.

(a) As noted, it will suffice to show that $A_1 \cup B_2$ is countable, where $B_2 = A_2 \setminus A_1$. Since A_1 is countable, there exists a bijection $f : \mathbf{N} \to A_1$. Consider the following cases.

Case 1. If B_2 is empty, then $A_1 \cup B_2 = A_1$, which is countable by assumption.

Case 2. Suppose that B_2 is non-empty and finite, say $B_2 = \{x_1, ..., x_k\}$ for some $k \in \mathbf{N}$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

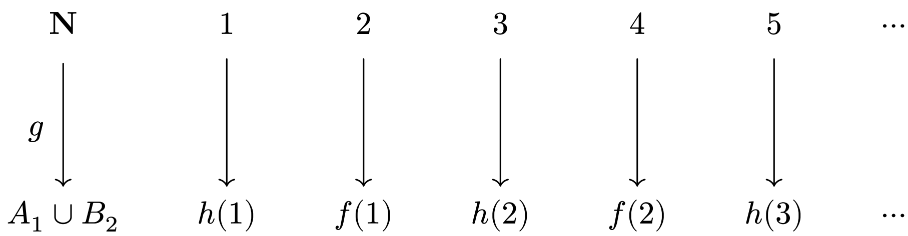
$$g(n) = \begin{cases} x_n & \text{if } 1 \le n \le k, \\ f(n-k) & \text{if } k < n. \end{cases}$$



The injectivity of g follows as A_1 and B_2 are disjoint and f is injective. For the surjectivity of g, it is clear that every element of B_2 belongs to the range of g; the surjectivity of f implies that the elements of A_1 belong to the range of g also.

Case 3. Suppose that B_2 is infinite. Since B_2 is a subset of the countable set A_2, Exercise 1.5.1 implies that B_2 is countable, i.e. there exists a bijection $h : \mathbf{N} \to B_2$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$g(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even,} \\ h\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$



To see that g is injective, suppose that m and n are distinct positive integers.

Case 3.1. If both of m and n are even then $g(m) \neq g(n)$ since f is injective.

Case 3.2. If both of m and n are odd then $g(m) \neq g(n)$ since h is injective.

Case 3.3. If one of m and n is even and the other is odd then $g(m) \neq g(n)$ since f maps into A_1, h maps into B_2, and $A_1 \cap B_2 = \emptyset$.

To see that g is surjective, let $x \in A_1 \cup B_2$ be given. Since $A_1 \cap B_2 = \emptyset$, exactly one of the statements $x \in A_1$ or $x \in B_2$ holds. Suppose $x \in A_1$. Because f is surjective, there is a positive integer n such that $f(n) = x$; it follows that $g(2n) = f(n) = x$. If $x \in B_2$, then the surjectivity of h implies that there is a positive integer n such that $h(n) = x$; it follows that $g(2n-1) = h(n) = x$. We may conclude that g is a bijection and hence that $A_1 \cup B_2$ is countable.

A simple induction argument proves the more general statement in Theorem 1.5.8 (i). Let $P(n)$ be the statement that for countable sets $A_1, \ldots, A_n$, the union $A_1 \cup \cdots \cup A_n$ is countable. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and suppose we have countable sets $A_1, \ldots, A_n, A_{n+1}$. Let $A' = A_1 \cup \cdots \cup A_n$; the induction hypothesis guarantees that A' is countable. Observe that

$$A_1 \cup \cdots \cup A_n \cup A_{n+1} = A' \cup A_{n+1}.$$

Since A' and A_{n+1} are countable, the union $A' \cup A_{n+1}$ is also countable by our previous proof, i.e. $P(n+1)$ holds. This completes the induction step and the proof.

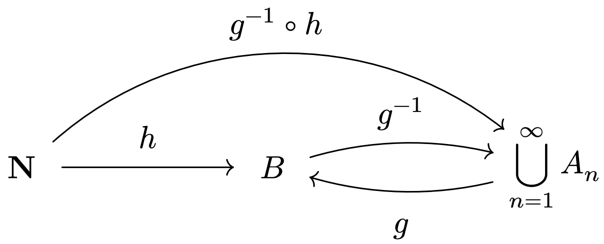
(b) Induction can only be used to show that a particular statement $P(n)$ holds for each value of $n \in \mathbf{N}$.

(c) For each $n \in \mathbf{N}$ there exists a bijection $f_n : \mathbf{N} \to A_n$. Let $a_{mn} = f_{n(m)}$ and arrange these into another two-dimensional array like so:

A_1	A_2	A_3	A_4	A_5	$\cdots$
a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\cdots$
a_{21}	a_{22}	a_{23}	a_{24}	$\ddots$	
a_{31}	a_{32}	a_{33}	$\ddots$		
a_{41}	a_{42}	$\ddots$			
a_{51}	$\ddots$				
$\vdots$					

1	3	6	10	15	$\cdots$
2	5	9	14	$\ddots$	
4	8	13	$\ddots$		
7	12	$\ddots$			
11	$\ddots$				
$\vdots$					

Since each f_n is surjective, each element of $\bigcup_{n=1}^{\infty} A_n$ appears somewhere in the left array. We define a function $g : \bigcup_{n=1}^{\infty} A_n \to \mathbf{N}$ by working through the grid along the diagonals (first a_{11}, then a_{22}, then a_{31}, and so on), mapping an element a_{mn} to the natural number appearing in the corresponding position in the right array. The A_n's may have elements in common; if we encounter an element a_{mn} that we have already seen before, we simply skip this element and move on to the next one. In this way, we obtain an injective function g. If we denote the range of g by $B \subseteq \mathbf{N}$, then $g : \bigcup_{n=1}^{\infty} A_n \to B$ is a bijection. Since the infinite set A_1 is contained in the union $\bigcup_{n=1}^{\infty} A_n$ and g is injective, it must be the case that B is infinite: Exercise 1.5.1 then implies that B is countable, i.e. there is a bijection $h : \mathbf{N} \to B$. It follows that the function $g^{-1} \circ h : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ is a bijection and we may conclude that $\bigcup_{n=1}^{\infty} A_n$ is countable.



Exercise 1.5.4.

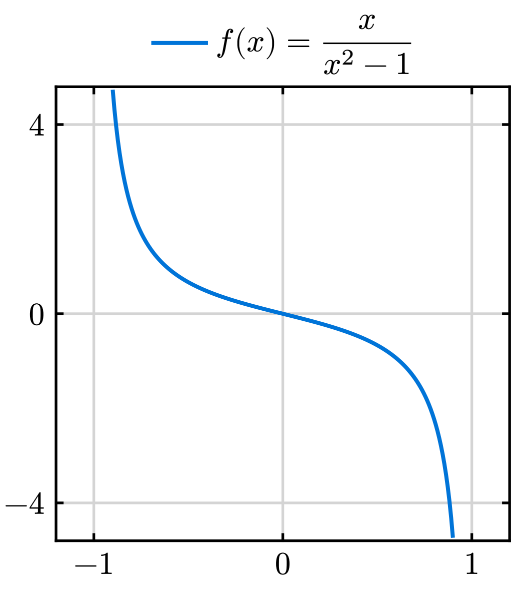
(a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

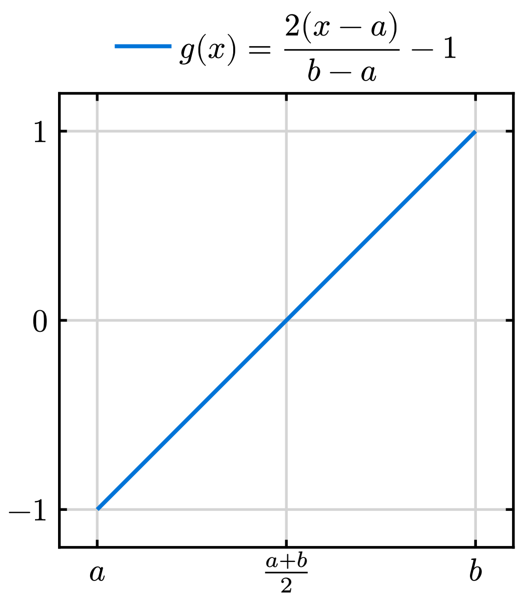
(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to product the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

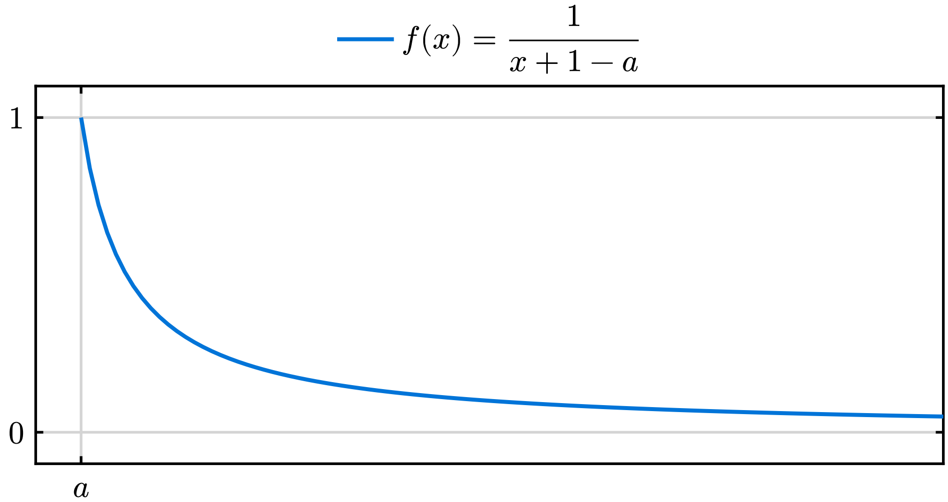
Solution.

(a) Let $f : (-1, 1) \to \mathbf{R}$ be the bijection given by $f(x) = \frac{x}{x^2-1}$ (see Example 1.5.4) and let $g : (a, b) \to (-1, 1)$ be given by $g(x) = \frac{2(x-a)}{b-a} - 1$; it is straightforward to verify that g is a bijection. Thus $(a, b) \sim (-1, 1) \sim \mathbf{R}$ and it follows from Exercise 1.5.5 that $(a, b) \sim \mathbf{R}$.





(b) The bijection $f : (a, \infty) \to (0, 1)$ given by $f(x) = \frac{1}{x+1-a}$ shows that $(a, \infty) \sim (0, 1)$. Since $(0, 1) \sim \mathbf{R}$ by part (a), Exercise 1.5.1 shows that $(a, \infty) \sim \mathbf{R}$.

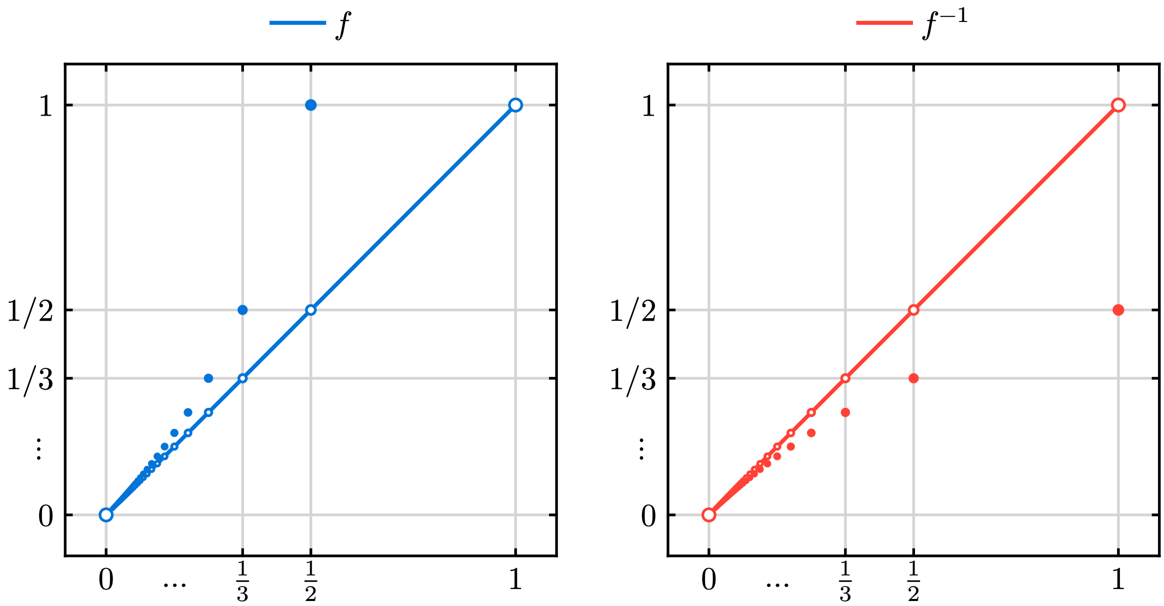


(c) Note that $[0,1) \sim (0,1]$ via the map $x \mapsto 1-x$ and so, by Exercise 1.5.5, it will suffice to show that $(0,1) \sim (0,1]$. Define a function $f : (0,1) \to (0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

This function is a bijection since it has an inverse $f^{-1} : (0,1] \to (0,1)$ given by

$$f^{-1}(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$



Exercise 1.5.5.

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Solution.

(a) The identity function $f : A \to A$ given by $f(x) = x$ is a bijection.

(b) Since $A \sim B$, there is a bijection $f : A \to B$. A function is bijective if and only if it has an inverse function $f^{-1} : B \to A$, which must also be bijective.

(c) There are bijections $f : A \to B$ and $g : B \to C$. It follows that the composition $g \circ f : A \to C$ is also a bijection.

Exercise 1.5.6.

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution.

(a) $\{(n, n+1) : n \in \mathbf{N}\}$ is a countable collection of disjoint open intervals.

(b) No such collection exists. If there was such a collection $\{I_a : a \in A\}$, for some uncountable set A, then using the density of $\mathbf{Q}$ in $\mathbf{R}$ we may choose a rational number $r_a \in I_a$ for each $a \in A$. Because the intervals are disjoint, each r_a must be distinct, i.e. the map $a \mapsto r_a$ is an injection. It follows that $\{r_a : a \in A\}$ is an uncountable subset of $\mathbf{Q}$—but this contradicts Theorem 1.5.6 (i) and Theorem 1.5.7.

Exercise 1.5.7. Consider the open interval $(0,1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0,1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to product a 1-1 function that maps S into $(0,1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999...)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0,1) \sim S$.

Solution.

(a) The map $f : (0,1) \to S$ given by $f(x) = \left(x, \frac{1}{2}\right)$ is injective.

(b) For $(x, y) \in S$, suppose x has decimal representation $0.x_1x_2x_3...$ and y has decimal representation $0.y_1y_2y_3...$, where if necessary we choose the decimal representation terminating in 0's. To define $g : S \to (0,1)$, let $g(x, y) = 0.x_1y_1x_2y_2x_3y_3...$

$$
\begin{array}{ccccccccccc}
x & = & 0 & . & x_1 & & x_2 & & x_3 & & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
g(x,y) & = & 0 & . & x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & \cdots \\
 & & & & & \uparrow & & \uparrow & & \uparrow & \\
y & = & 0 & . & & y_1 & & y_2 & & y_3 & \cdots
\end{array}
$$

For the injectivity of g, suppose we have $(x, y) \neq (a, b)$ in S, so that at least one of $x \neq a$ or $y \neq b$ holds. Assuming $x \neq a$ (the case where $y \neq b$ is handled similarly), let $0.x_1x_2x_3...$ be the decimal representation of x and let $0.a_1a_2a_3...$ be the decimal representation of a. Since $x \neq a$, there must be some index n such that $x_n \neq a_n$. If $g(x, y)$ has decimal representation $0.s_1 s_2 s_3...$ and $g(a, b)$ has decimal representation $0.t_1 t_2 t_3...$, then

$$s_{2n-1} = x_n \neq a_n = t_{2n-1}.$$

This implies that $g(x, y) \neq g(a, b)$, provided it is not the case that $g(x, y)$ terminates in 0's and $g(a, b)$ terminates in 9's, or vice versa. To rule this out, note that $g(a, b)$ terminates in 9's only if both a and b terminate in 9's—but our construction specifically chooses the decimal representations for a and b terminating in 0's if necessary. The case where $g(x, y)$ terminates in 9's is handled similarly.

This function g is not surjective since 0.1 does not belong to the range of g. Indeed,

$$g(x, y) = 0.x_1 y_1 x_2 y_2 ... = 0.1000...$$

implies that $y = 0$, but $(x, 0) \notin S$ for any $x \in (0, 1)$.

Exercise 1.5.8. Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution. Suppose $a \in (0, 1]$; we claim that $B \cap (a, 2]$ must be a (possibly empty) finite set. By the Archimedean Property (Theorem 1.4.2), there is an $n \in \mathbf{N}$ such that $na > 2$. If $B \cap (a, 2]$ contains at least n elements, say $\{b_1, ..., b_n\}$, then since each $b_i > a$ we have

$$b_1 + \cdots + b_n > na > 2.$$

This contradicts our hypotheses, so it must be the case that $B \cap (a, 2]$ contains less than n elements. Our claim follows.

Any element of B must be less than or equal to 2, so $B \subseteq (0, 2]$ and it follows that

$$B = \bigcup_{n=1}^{\infty} \left(B \cap \left(\tfrac{1}{n}, 2 \right] \right).$$

By our previous paragraph, each $B \cap \left(\frac{1}{n}, 2 \right]$ is a finite set. Thus the expression above shows that B is a countable union of finite sets and hence, by Theorem 1.5.8, B is either finite or countable.

Exercise 1.5.9. A real number $x \in \mathbf{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \dots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3}+\sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution.

(a) $\sqrt{2}$ is a root of the polynomial x^2-2, $\sqrt[3]{2}$ is a root of the polynomial x^3-2, and $\sqrt{3}+\sqrt{2}$ is a root of the polynomial x^4-10x^2+1.

(b) We will use the following useful corollary of Theorem 1.5.8 (ii).

Lemma L.5. If $A_1, \dots, A_n$ are countable sets, then $A_1 \times \cdots \times A_n$ is also countable.

Proof. Suppose that A and B are countable sets, so that $B = \{b_1, b_2, b_3, \dots\}$. For each $n \in \mathbf{N}$, it is clear that the set $A \times \{b_n\}$ is countable. Now observe that

$$A \times B = \bigcup_{n=1}^{\infty} (A \times \{b_n\}).$$

It follows from Theorem 1.5.8 (ii) that $A \times B$ is countable. A straightforward induction argument proves the general case. □

Let P_n be the collection of polynomials with integer coefficients that have degree n, i.e. $P_n = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_n, \dots, a_0 \in \mathbf{Z}, a_n \neq 0\}$. Notice that

$$P_n \sim (\mathbf{Z} \setminus \{0\}) \times \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{n \text{ times}}$$

via the map

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mapsto (a_n, a_{n-1}, \dots, a_1, a_0).$$

It follows from Lemma L.5 that P_n is countable. For a polynomial $p \in P_n$, let R_p be the set of its roots, i.e. $R_p = \{x \in \mathbf{R} : p(x) = 0\}$, and note that R_p is always a finite set. Now observe that

$$A_n = \bigcup_{p \in P_n} R_p,$$

demonstrating that A_n is a countable union of finite sets; it follows from Theorem 1.5.8 that A_n is either finite or countable. Since $\sqrt[n]{k} \in A_n$ for each $k \in \mathbf{N}$ (it is a root of the polynomial $x^n - k$), we see that A_n must be infinite and hence countable.

(c) If we let A be the set of all algebraic numbers then $A = \bigcup_{n=1}^{\infty} A_n$, i.e. A is a countable union of countable sets. It follows from Theorem 1.5.8 (ii) that A is countable.

A consequence of this is that the set of transcendental numbers A^c must be uncountable. To see this, note that $\mathbf{R} = A \cup A^c$, the union of two countable sets is countable, and $\mathbf{R}$ is not countable.

Exercise 1.5.10.

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and let $\alpha = \sup A$. Is $C \cap [\alpha,1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution.

(a) If we suppose that for each $a \in (0,1)$ the set $C \cap [a,1]$ is countable, then we can express C as a countable union of countable sets:

$$C = (C \cap \{0\}) \cup \bigcup_{n=2}^{\infty} \left(C \cap \left[\tfrac{1}{n}, 1\right]\right).$$

This implies that C is countable (Theorem 1.5.8 (ii)). Thus, given that C is uncountable, there must exist some $a \in (0,1)$ such that $C \cap [a,1]$.

(b) Not necessarily. If $C = [0,1]$, then for all $a \in (0,1)$ we have $C \cap [a,1] = [a,1]$, which is uncountable. Thus $A = (0,1)$, so that $\alpha = 1$, but $C \cap [\alpha,1] = \{1\}$ is not uncountable.

(c) The statement is no longer true in general. If we let $C = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ then no matter which $a \in (0,1)$ we choose, the intersection $C \cap [a,1]$ is a finite set.

Exercise 1.5.11 (Schröder-Bernstein Theorem). Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution.

(a) Abusing notation slightly, we have bijections $f : A \to B$ and $g : B' \to A'$, and their inverses $f^{-1} : B \to A$ and $g^{-1} : A' \to B'$. Since $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, the functions $h : X \to Y$ and $h' : Y \to X$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g^{-1}(x) & \text{if } x \in A', \end{cases} \qquad h'(y) = \begin{cases} f^{-1}(y) & \text{if } y \in B, \\ g(y) & \text{if } y \in B' \end{cases}$$

are well-defined. It is straightforward to verify that h and h' are mutual inverses and thus $X \sim Y$.

(b) If A_1 is empty then $X = g(Y)$, i.e. g is surjective. Since g is injective by assumption, it immediately follows that $X \sim Y$ via g.

Let $P(n)$ be the statement that $\{A_1, ..., A_n\}$ is a pairwise disjoint collection of sets; to prove that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection, we will first use induction to prove that $P(n)$ holds for all $n \in \mathbf{N}$. The truth of $P(1)$ is clear, so suppose that $P(n)$ holds for some $n \in \mathbf{N}$. To demonstrate the truth of $P(n+1)$, we need to show that $A_k \cap A_{n+1} = \emptyset$ for all $1 \le k \le n$. Because $A_{n+1} = g(f(A_n)) \subseteq g(Y)$ and $A_1 = X \setminus g(Y)$, we see that $A_1 \cap A_{n+1} = \emptyset$. If $n \ge 2$, suppose that $2 \le k \le n$ and observe that

$$\begin{aligned} A_k \cap A_{n+1} &= g(f(A_{k-1})) \cap g(f(A_n)) \\ &= g(f(A_{k-1} \cap A_n)) && (f \text{ and } g \text{ are injective}) \\ &= g(f(\emptyset)) && (\text{induction hypothesis}) \\ &= \emptyset. \end{aligned}$$

Hence $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

It is now straightforward to show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of sets. Let A_m and A_n be given and suppose without loss of generality that $m < n$. By the previous paragraph the collection $\{A_1, ..., A_m, ..., A_n\}$ is pairwise disjoint and thus $A_m \cap A_n = \emptyset$.

That $\{f(A_n) : n \in \mathbf{N}\}$ is a pairwise disjoint collection now follows immediately from the injectivity of f.

(c) Observe that

$$f(A) = f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n) = B,$$

where we have used that the image of a union is the union of the images; the proof of this is similar to the proof of the special case given in Exercise 1.2.7 (d).

(d) Notice that

$$\begin{aligned} b \in B' \quad &\Leftrightarrow \quad b \notin f(A_n) \text{ for all } n \in \mathbf{N} \\ &\Leftrightarrow \quad g(b) \notin g(f(A_n)) \text{ for all } n \in \mathbf{N} \qquad (g \text{ is injective}) \\ &\Leftrightarrow \quad g(b) \notin A_{n+1} \text{ for all } n \in \mathbf{N} \\ &\Leftrightarrow \quad g(b) \notin A_n \text{ for all } n \geq 2. \end{aligned}$$

Notice further that $g(y) \notin X \setminus g(Y) = A_1$ for any $y \in Y$. It follows that

$$b \in B' \quad \Leftrightarrow \quad g(b) \notin A_n \text{ for all } n \in \mathbf{N} \quad \Leftrightarrow \quad g(b) \in A'. \qquad (*)$$

Thus g maps B' into A'. To see that $g : B' \to A'$ is surjective, observe that for any $a \in A'$ we have, in particular, $a \notin A_1 = X \setminus g(Y)$, so that $a \in g(Y)$, i.e. $a = g(y)$ for some $y \in Y$. It then follows from $(*)$ that $y \in B'$.

1.6. Cantor's Theorem

Exercise 1.6.1. Show that $(0,1)$ is uncountable if and only if $\mathbf{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

Solution. We have $(0,1) \sim \mathbf{R}$ by Exercise 1.5.4 (a).

Exercise 1.6.2.

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 \ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

Solution.

(a) We have decimal expansions

$$f(1) = 0.a_{11}a_{12}a_{13}a_{14}\ldots \quad \text{and} \quad x = 0.b_1 b_2 b_3 b_4 \ldots$$

By construction, $b_1 \neq a_{11}$. This implies that $f(1) \neq x$, provided these decimal expansions are not two different expansions of the same real number (for example, 0.3 and 0.2999...). However, since the only way this can occur is when one decimal expansion terminates in repeating 0's and the other terminates in repeating 9's, and the digits b_n are always either 2 or 3, we see that $0.b_1 b_2 b_3 b_4 \ldots$ must be the unique decimal representation of x.

(b) Since $0.b_1 b_2 b_3 b_4 \ldots$ is the unique decimal expansion of the real number x and $b_n \neq a_{nn}$, we have $x \neq f(n)$ for every $n \in \mathbf{N}$. Here is an example construction of x given some function $f : \mathbf{N} \to (0,1)$:

$$\begin{array}{ccccccccccc}
f(1) & = & 0 & . & 9 & 2 & 8 & 4 & 7 & 6 & \ldots \\
f(2) & = & 0 & . & 2 & 2 & 8 & 4 & 9 & 1 & \ldots \\
f(3) & = & 0 & . & 9 & 9 & 1 & 0 & 2 & 5 & \ldots \\
f(4) & = & 0 & . & 2 & 1 & 1 & 9 & 2 & 1 & \ldots \\
f(5) & = & 0 & . & 1 & 2 & 5 & 7 & 2 & 3 & \ldots \\
f(6) & = & 0 & . & 9 & 7 & 7 & 5 & 1 & 8 & \ldots \\
\vdots & & & & & & & & & & \\
x & = & 0 & . & 2 & 3 & 2 & 2 & 3 & 2 & \ldots
\end{array}$$

Notice how the first digit (after the decimal point) of x differs from the first digit of $f(1)$, the second digit of x differs from the second digit of $f(2)$, and so on.

(c) The real number x belongs to $(0,1)$ but not to the image of f, which contradicts our assumption that f was surjective. It follows that there cannot exist a bijection between $\mathbf{N}$ and $(0,1)$. Since $(0,1)$ is infinite, we may conclude that $(0,1)$ is uncountable.

Exercise 1.6.3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, 1/2 can also be written as .5 or .4999... Doesn't this cause some problems?

Solution.

(a) The problem with this reasoning is that the real number

$$x = 0.b_1 b_2 b_3 b_4 \dots$$

that we construct may not be rational. For example, consider the function $f : \mathbf{N} \to (0,1) \cap \mathbf{Q}$ given by

$$\begin{array}{rclrcll} f(1) &=& 0.3 & f(6) &=& 0.000003 & \\ f(2) &=& 0.02 & f(7) &=& 0.0000003 & \\ f(3) &=& 0.003 & f(8) &=& 0.00000003 & \cdots \\ f(4) &=& 0.0003 & f(9) &=& 0.000000002 & \\ f(5) &=& 0.00002 & f(10) &=& 0.0000000003 & \end{array}$$

This results in $x = 0.2322322232\dots$, which is not rational since its decimal expansion does not repeat. So while x does not belong to the image of f, this is not a problem because x does not belong to $(0,1) \cap \mathbf{Q}$ either.

(b) We addressed this issue in Exercise 1.6.2 (a).

Exercise 1.6.4. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1,0,1,0,1,0,1,0,\dots)$ is an element of S, as is the sequence $(1,1,1,1,1,1,\dots)$.

Give a rigorous argument showing that S is uncountable.

Solution. Suppose we have a function $f : \mathbf{N} \to S$. For each $m \in \mathbf{N}$, let a_{mn} be the element in the n^{th} position of $f(m)$, so that

$$f(m) = (a_{m1}, a_{m2}, a_{m3}, a_{m4}, \ldots) \in S.$$

Let $b = (b_1, b_2, b_3, b_4, \ldots)$ be the sequence given by

$$b_n = \begin{cases} 0 & \text{if } a_{nn} = 1, \\ 1 & \text{if } a_{nn} = 0. \end{cases}$$

Notice that $b \in S$ but $b \neq f(n)$ for any $n \in \mathbf{N}$, since b differs from $f(n)$ in the n^{th} position. Here is an example construction of the sequence b, given some $f : \mathbf{N} \to S$:

$$\begin{array}{rcl} f(1) & = & (\color{red}{1}, \ 0, \ 0, \ 1, \ 0, \ 1, \ \ldots \\ f(2) & = & (0, \ \color{red}{0}, \ 1, \ 1, \ 1, \ 0, \ \ldots \\ f(3) & = & (0, \ 1, \ \color{red}{1}, \ 0, \ 0, \ 0, \ \ldots \\ f(4) & = & (1, \ 1, \ 1, \ \color{red}{1}, \ 0, \ 0, \ \ldots \\ f(5) & = & (0, \ 0, \ 1, \ 0, \ \color{red}{0}, \ 1, \ \ldots \\ f(6) & = & (1, \ 0, \ 0, \ 1, \ 0, \ \color{red}{1}, \ \ldots \\ \vdots & & \\ b & = & (0, \ 1, \ 0, \ 0, \ 1, \ 0, \ \ldots \end{array}$$

Notice that b differs from $f(1)$ in the first position, from $f(2)$ in the second position, and so on.

Thus $b \notin f(\mathbf{N})$, so that f is not a surjection. Since f was arbitrary, it follows that there can be no bijection between $\mathbf{N}$ and S. Certainly S is infinite, so we may conclude that S is uncountable.

Exercise 1.6.5.

(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

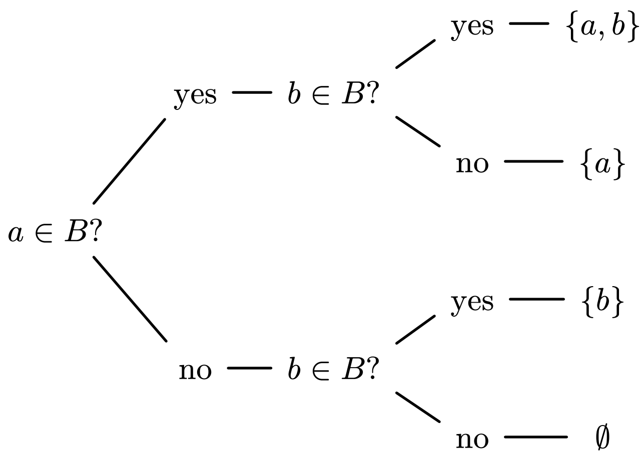
(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution.

(a) We have

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

(b) To form a subset B of A, for each element $a \in A$ we must decide whether to include a in B or not. This is a binary choice to be made for each of the n elements of A; it follows that there are 2^n subsets of A. For example, here is a tree listing all $2^2 = 4$ subsets of $\{a, b\}$:



Exercise 1.6.6.

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

Solution.

(a) Here are two injections $f : A \to P(A)$ and $g : A \to P(A)$:

$$\begin{aligned} f(a) &= \{a\}, & g(a) &= \{a, b\}, \\ f(b) &= \{b\}, & g(b) &= \{b, c\}, \\ f(c) &= \{c\}, & g(c) &= \{a, c\}. \end{aligned}$$

(b) Let g be given by

$$\begin{aligned} g(1) &= \{1\}, & g(3) &= \{3\}, \\ g(2) &= \{2\}, & g(4) &= \{4\}. \end{aligned}$$

(c) The power set of a finite set A always contains strictly more elements than A (Exercise 1.6.5 (b)). For finite sets, it is impossible to construct a surjective function from a set A to a set B if B contains strictly more elements than A.

Exercise 1.6.7. Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

Solution. For all three functions from Exercise 1.6.6 we have $B = \emptyset$, which does not belong to the range of any of the functions.

Exercise 1.6.8.

(a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution.

(a) and (b). We have $a' \in B$ if and only if $a' \notin f(a') = B$, which is a contradiction since a' either does or does not belong to B.

Exercise 1.6.9. Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution. First, let us show that $P(\mathbf{N}) \sim S$, where S is the set of all binary sequences defined in Exercise 1.6.4. Consider the function $f : P(\mathbf{N}) \to S$ given by $f(E) = (a_1, a_2, a_3, \ldots)$ where

$$a_n = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \notin E. \end{cases}$$

For example, $f(\{1,3,4,6,7,10\}) = (1,0,1,1,0,1,1,0,0,1,0,0,0,\ldots)$.

This function is a bijection since it has an inverse $f^{-1} : S \to P(\mathbf{N})$ given by

$$f^{-1}(a_1, a_2, a_3, \ldots) = \{n \in \mathbf{N} : a_n = 1\}.$$

Now let us show that $S \sim (0,1)$. Consider the function $g : S \to (0,1)$ given by

$$g(a_1, a_2, a_3, \ldots) = 0.5a_1a_2a_3\ldots,$$

where $0.5a_1a_2a_3\ldots$ is a decimal expansion (for example, $g(1,0,1,0,0,0,\ldots) = 0.5101$). This function is injective since if $a = (a_1, a_2, a_3, \ldots) \neq b = (b_1, b_2, b_3, \ldots)$, then there must exist some $n \in \mathbf{N}$ such that $a_n \neq b_n$. It follows that $g(a) \neq g(b)$, provided $g(a) = 0.5a_1a_2a_3\ldots$ and $g(b) = 0.5b_1b_2b_3\ldots$ are not two different decimal expansions of the same real number. This cannot be the case since each a_i and b_i is either 0 or 1, and never 9.

Now consider the function $h : (0,1) \to S$ given by

$$h(a) = h(0.a_1a_2a_3\ldots) = (a_1, a_2, a_3, \ldots),$$

where $0.a_1a_2a_3\ldots$ is the **binary** expansion of $a \in (0,1)$, choosing that expansion which terminates in 0's if a has two different binary expansions. This function is injective since if $a = 0.a_1a_2a_3\ldots \neq b = 0.b_1b_2b_3\ldots$, then there must be some $n \in \mathbf{N}$ such that $a_n \neq b_n$. It follows that $h(a) \neq h(b)$.

The Schröder-Bernstein Theorem (Exercise 1.5.11) now implies that $S \sim (0,1)$. We showed in Exercise 1.5.4 that $(0,1) \sim \mathbf{R}$ and thus

$$P(\mathbf{N}) \sim S \sim (0,1) \sim \mathbf{R}.$$

In Exercise 1.5.5 we showed that $\sim$ is an equivalence relation, so the chain of equivalences above allows us to conclude that $P(\mathbf{N}) \sim \mathbf{R}$.

Exercise 1.6.10. As a final exercise, answer each of the following by establishing a 1-1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0,1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0,1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution.

(a) Let $\mathbf{N}^{\{0,1\}}$ be the set of all functions from $\{0,1\}$ to $\mathbf{N}$. Consider the function $F : \mathbf{N}^{\{0,1\}} \to \mathbf{N} \times \mathbf{N}$ given by $F(f) = (f(0), f(1))$. This function is a bijection since it has an inverse $F^{-1} : \mathbf{N} \times \mathbf{N} \to \mathbf{N}^{\{0,1\}}$ given by $F^{-1}(a,b) = f$, where $f : \{0,1\} \to \mathbf{N}$ is the function satisfying $f(0) = a, f(1) = b$. Thus

$$\mathbf{N}^{\{0,1\}} \sim \mathbf{N} \times \mathbf{N} \sim \mathbf{N},$$

where we have used Lemma L.5 for the second equivalence. We may conclude that $\mathbf{N}^{\{0,1\}}$ is countable.

(b) The set of all functions from $\mathbf{N}$ to $\{0,1\}$ is nothing but the set S of all binary sequences defined in Exercise 1.6.4, since a function $f : \mathbf{N} \to \{0,1\}$ can be identified with the sequence $(f(0), f(1), f(2), \ldots)$. Thus the set of all functions from $\mathbf{N}$ to $\{0,1\}$ is uncountable, since we showed that S is uncountable in Exercise 1.6.4.

(c) We will construct an uncountable antichain contained in $P(\mathbf{N})$. Let p_m be the m^{th} prime number, i.e. $(p_1, p_2, p_3, p_4, \ldots) = (2, 3, 5, 7, \ldots)$, and note that by the fundamental theorem of arithmetic the map

$$\begin{array}{rcl} \mathbf{N} \times \mathbf{N} & \to & \mathbf{N} \\ (m,n) & \mapsto & p_m^n \end{array}$$

is injective. Define a map $\Psi : P(\mathbf{N}) \setminus \{\emptyset, \mathbf{N}\} \to P(\mathbf{N})$ by

$$\Psi(X) = \{p_m^n : m \in X, n \notin X\}.$$

Let $\mathcal{A} \subseteq P(\mathbf{N})$ be the image of Ψ and let $X \neq Y$ in $P(\mathbf{N}) \setminus \{\emptyset, \mathbf{N}\}$ be given. Observe that

$$\begin{array}{rcl} X \neq \emptyset & \Rightarrow & \text{there is some } \ell \in X, \\ Y \neq \emptyset \text{ and } Y \neq X & \Rightarrow & \text{there is some } m \in Y \text{ such that } m \notin X, \\ Y \neq \mathbf{N} & \Rightarrow & \text{there is some } n \notin Y. \end{array}$$

It follows that $p_\ell^m \in \Psi(X) \setminus \Psi(Y)$ and $p_m^n \in \Psi(Y) \setminus \Psi(X)$, so that $\Psi(X)$ is not contained in $\Psi(Y)$ and $\Psi(Y)$ is not contained in $\Psi(X)$. This demonstrates both that the map Ψ is injective and that $\mathcal{A}$ is an antichain. Since $P(\mathbf{N}) \setminus \{\emptyset, \mathbf{N}\}$ is uncountable (Exercise 1.6.9), it follows that $\mathcal{A} \subseteq P(\mathbf{N})$ is an uncountable antichain.

Chapter 2. Sequences and Series

2.2. The Limit of a Sequence

Exercise 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \varepsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution. First observe that, by taking $N = 1$ in the first statement,

$$(\text{for all } N \in \mathbf{N},\ n \geq N \quad \Rightarrow \quad |x_n - x| < \varepsilon) \quad \Leftrightarrow \quad (\text{for all } n \in \mathbf{N},\ |x_n - x| < \varepsilon).$$

Thus a sequence verconges to x if there exists an $\varepsilon > 0$ such that $|x_n - x| < \varepsilon$ for all $n \in \mathbf{N}$, or equivalently such that $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \in \mathbf{N}$.

For an example of a vercongent sequence that diverges, consider $(x_n) = (1, 0, 1, 0, ...)$. This sequence verconges to $\frac{1}{2}$ since $\left|x_n - \frac{1}{2}\right| = \frac{1}{2} < 1$ for all $n \in \mathbf{N}$. Now suppose that $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$, so that there is some $N \in \mathbf{N}$ such that $|x_n - x| < \frac{1}{2}$ whenever $n \geq N$, and observe that

$$1 = \left|x_N - x_{N+1}\right| \leq |x_N - x| + \left|x_{N+1} - x\right| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e. $1 < 1$. It follows that (x_n) does not converge to any $x \in \mathbf{R}$.

A sequence can verconge to two different values. For example, the sequence $(x_n) = (0, 0, 0, ...)$ verconges to both 0 and 1:

$$|x_n| = 0 < 1 \text{ for all } n \in \mathbf{N} \quad \text{and} \quad |x_n - 1| = 1 < 2 \text{ for all } n \in \mathbf{N}.$$

This definition of "vercongence" describes the bounded sequences (see Definition 2.3.1): a sequence which verconges to some $x \in \mathbf{R}$ must be bounded and conversely any bounded sequence verconges to some $x \in \mathbf{R}$.

Exercise 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

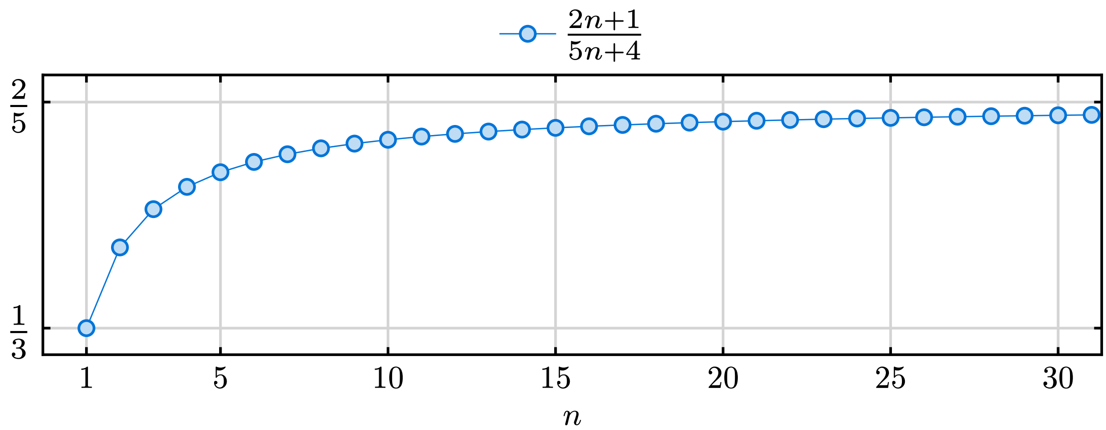
(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution.

(a) Let $\varepsilon > 0$ be given. Using the Archimedean Property (Theorem 1.4.2), let $N \in \mathbf{N}$ be such that $N > \frac{3}{25\varepsilon}$ and observe that for $n \geq N$ we have

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \frac{3}{25n+20} < \frac{3}{25n} \leq \frac{3}{25N} < \varepsilon.$$

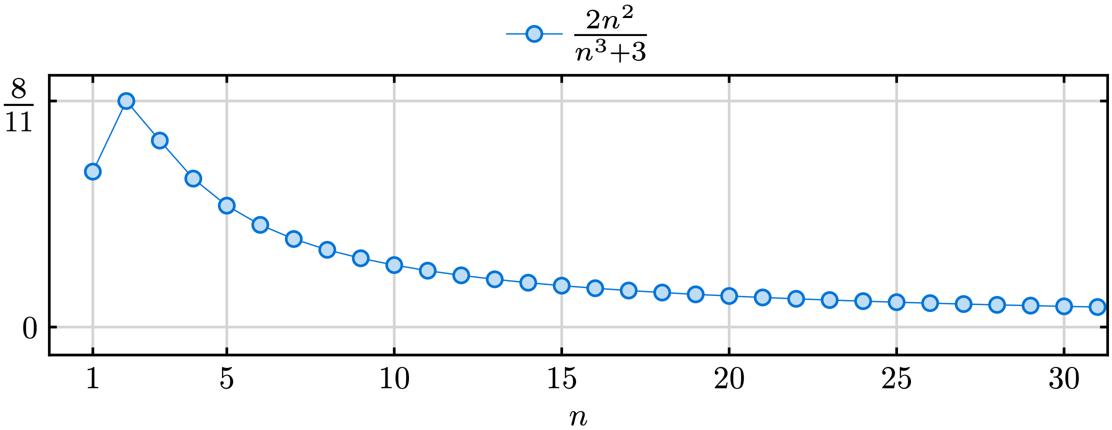
Thus $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.



(b) Let $\varepsilon > 0$ be given. Using the Archimedean Property (Theorem 1.4.2), let $N \in \mathbf{N}$ be such that $N > \frac{2}{\varepsilon}$ and observe that for $n \geq N$ we have

$$\left| \frac{2n^2}{n^3+3} \right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

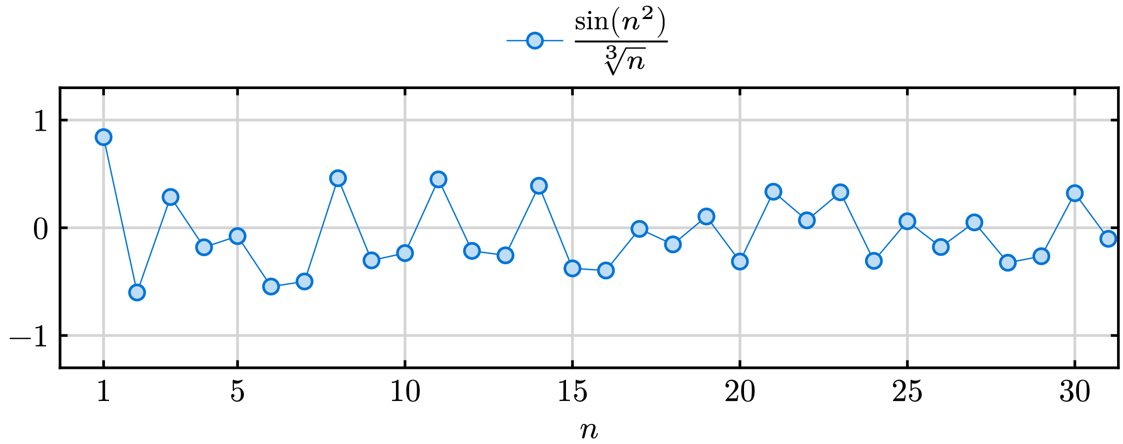
It follows that $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$.



(c) Let $\varepsilon > 0$ be given. Using the Archimedean Property (Theorem 1.4.2), let $N \in \mathbf{N}$ be such that $N > \frac{1}{\varepsilon^3}$ and observe that for $n \geq N$ we have

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \varepsilon.$$

It follows that $\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.



Exercise 2.2.3. Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution.

(a) We would have to find a college in the United States where every student is less than seven feet tall.

(b) We would have to find a college in the United States where each professor gives at least one student a grade of C or worse.

(c) We would have to show that every college in the United States has a student who is less than six feet tall.

Exercise 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$, it is possible to find n consecutive ones somewhere in the sequence.

Solution.

(a) Consider the sequence $(1, 0, 1, 0, \ldots)$. This sequence has an infinite number of ones but, as shown in Exercise 2.2.1, diverges.

(b) This is impossible. Suppose (x_n) is such a sequence with $\lim_{n\to\infty} x_n = x \neq 1$. There then exists some $N \in \mathbf{N}$ such that $|x_n - x| < |1 - x|$ whenever $n \geq N$. Because this sequence contains infinitely many ones, there must be some $m \geq N$ such that $x_m = 1$ —but this implies that $|x_m - x| = |1 - x| < |1 - x|$, which is a contradiction.

(c) Consider the sequence

$$(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots).$$

For each $n \in \mathbf{N}$ let $m = \frac{n(n+1)}{2}$ and note that we can find n consecutive ones starting at the m^{th} position and, for $n \geq 2$, we can find a zero at the $(m-1)^{\text{th}}$ position. Furthermore, the sequence is divergent. If $x \in \mathbf{R}$ is such that $\lim_{n\to\infty} x_n = x$, then there must be some $N \in \mathbf{N}$ such that $|x_n - x| < \frac{1}{2}$ whenever $n \geq N$. Because the sequence contains infinitely many ones and zeros, we can find indices $k, \ell \geq N$ such that $x_k = 1$ and $x_\ell = 0$. It follows that

$$1 = |x_k - x_\ell| \leq |x_k - x| + |x_\ell - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e. $1 < 1$. Thus (x_n) does not converge to any $x \in \mathbf{R}$.

Exercise 2.2.5. Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

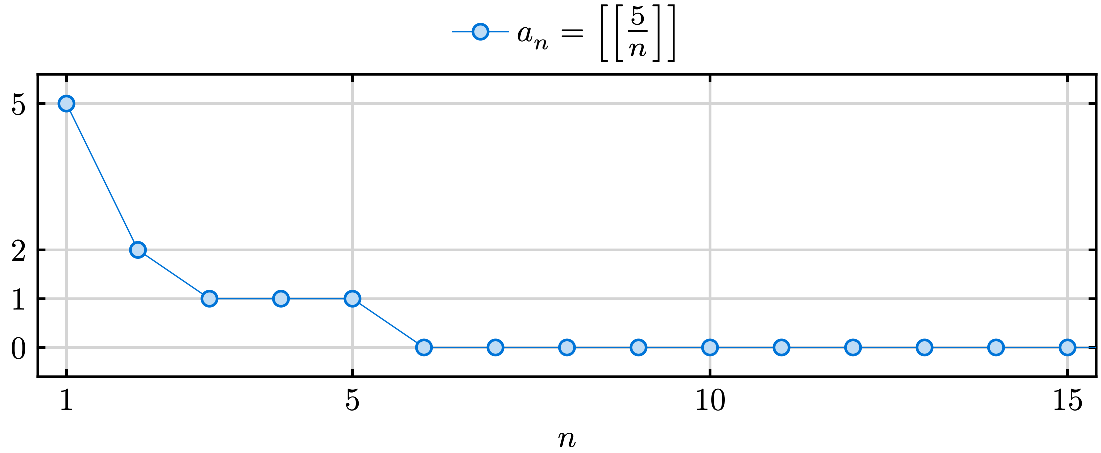
Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ε-neighborhood, the larger N may have to be."

Solution.

(a) Observe that

$$n \geq 6 \quad \Rightarrow \quad 0 < \frac{5}{n} < 1 \quad \Rightarrow \quad a_n = \left[\left[\frac{5}{n}\right]\right] = 0.$$

So for any $\varepsilon > 0$, if we take $N = 6$ then $|a_n| < \varepsilon$ for all $n \geq N$; it follows that $\lim_{n\to\infty} a_n = 0$.



(b) We claim that $\lim_{n\to\infty} a_n = 1$. Observe that

$$n \geq 7 \quad \Rightarrow \quad \frac{1}{n} < \frac{1}{6} \quad \Rightarrow \quad \frac{4}{n} < \frac{2}{3} \quad \Rightarrow \quad \frac{4}{n} + \frac{1}{3} < 1.$$

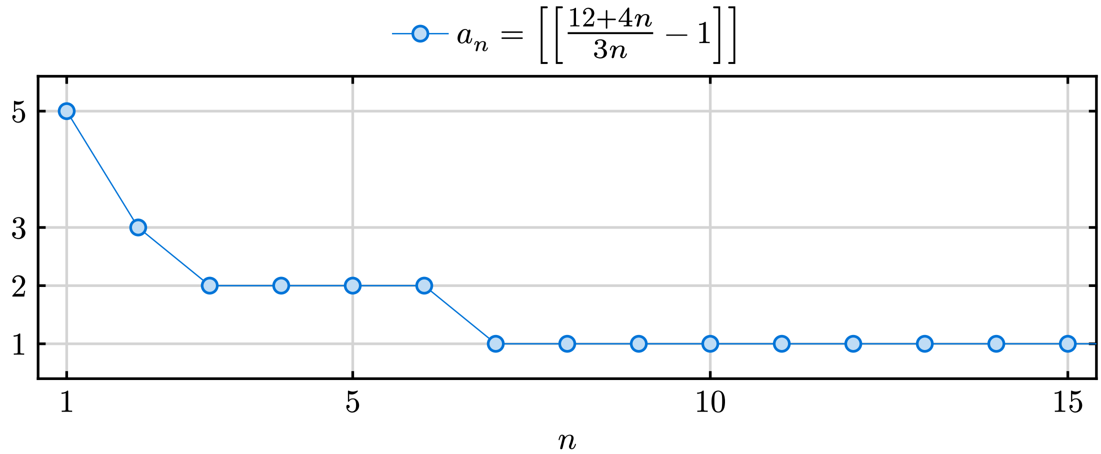
Hence for $n \geq 7$ we have

$$0 < \frac{4}{n} + \frac{1}{3} < 1 \quad \Rightarrow \quad \left[\left[\frac{4}{n} + \frac{1}{3}\right]\right] = 0.$$

So for any $\varepsilon > 0$, if we take $N = 7$ then

$$n \geq N \quad \Rightarrow \quad [[a_n - 1]] = \left[\left[\frac{12 + 4n}{3n} - 1\right]\right] = \left[\left[\frac{4}{n} + \frac{1}{3}\right]\right] = 0 < \varepsilon.$$

Thus $\lim_{n\to\infty} a_n = 1$.



These examples demonstrate that taking smaller ε-neighbourhoods may not require us to take larger values of N; the same value of N in each example works for every ε-neighbourhood that we choose.

Exercise 2.2.6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Solution. Let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \quad \Rightarrow \quad |a_n - a| < \frac{\varepsilon}{2} \quad \text{and} \quad n \geq N_2 \quad \Rightarrow \quad |a_n - b| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$|a - b| = |a - a_n + a_n - b| \leq |a_n - a| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $|a - b| < \varepsilon$ for any $\varepsilon > 0$; it follows from Theorem 1.2.6 that $a = b$.

Exercise 2.2.7. Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution.

(a) The sequence $(-1)^n$ is frequently but not eventually in the set $\{1\}$. To see this, let $N \in \mathbf{N}$ be given. If N is even, then $(-1)^N \in \{1\}$ and $(-1)^{N+1} \notin \{1\}$, and if N is odd then $(-1)^N \notin \{1\}$ and $(-1)^{N+1} \in \{1\}$. In any case, we can always find indices $m, n \geq N$ such that $(-1)^m \notin \{1\}$ (this shows that the sequence is not eventually in $\{1\}$) and such that $(-1)^n \in \{1\}$ (this shows that the sequence is frequently in $\{1\}$).

(b) Eventually is the stronger definition. Frequently does not imply eventually, as part (a) shows, but eventually does imply frequently. To see this, suppose that (a_n) is eventually in a set A, i.e. there is an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$. Let $M \in \mathbf{N}$ be given, let $n = \max\{M, N\}$, and observe that $n \geq M$ and $a_n \in A$. It follows that (a_n) is frequently in A.

(c) The term we want is eventually. Here is a rephrasing of Definition 2.2.3B: a sequence (a_n) converges to a if, given any $\varepsilon > 0$, the sequence (a_n) is eventually in the ε-neighbourhood $V_\varepsilon(a)$ of a.

(d) Such a sequence is not necessarily eventually in $(1.9, 2.1)$. For example, consider the sequence $(x_n) = (2, 0, 2, 0, 2, ...)$. For any $N \in \mathbf{N}$, we can always find an index $n \geq N$ (either $n = N$ or $n = N + 1$) such that $x_n = 0 \notin (1.9, 2.1)$. However, such a sequence must be frequently in $(1.9, 2.1)$. Indeed, for any $N \in \mathbf{N}$ there must exist an index $n \geq N$ such that $x_n = 2 \in (1.9, 2.1)$, otherwise there would be only finitely many twos in the sequence.

Exercise 2.2.8. For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, ...)$ zero-heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if ....

Solution.

(a) This sequence is zero-heavy: $M = 1$ works. Indeed, let $N \in \mathbf{N}$ be given. If N is odd then let $n = N$ and if N is even then let $n = N + 1$. In either case we have $N \leq n \leq N + 1$ and $x_n = 0$.

(b) A zero-heavy sequence must contain an infinite number of zeros. To see this, suppose (x_n) is a sequence with a finite number of zeros, i.e. there is an $N \in \mathbf{N}$ such that $x_n \neq 0$ for all $n \geq N$. It follows that, no matter which M we choose, we will never be able to find $n \in \mathbf{N}$ with $N \leq n \leq N + M$ and $x_n = 0$. Thus the sequence (x_n) is not zero-heavy.

(c) A sequence with an infinite number of zeros is not necessarily zero-heavy. For a counterexample, consider the sequence

$$(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, ...).$$

This sequence contains infinitely many zeros, but is not zero-heavy. To see this, let $M \in \mathbf{N}$ be given. It is always possible to find M consecutive ones in the sequence (see Exercise 2.2.4 (c)); suppose this string of ones starts at $x_N = 1$. It follows that for each $n \in \mathbf{N}$ satisfying $N \leq n \leq N + M$ we have $x_n = 1 \neq 0$. Thus (x_n) is not zero-heavy.

(d) A sequence is *not* zero-heavy if for every $M \in \mathbf{N}$ there exists an $N \in \mathbf{N}$ such that $x_n \neq 0$ for each $n \in \mathbf{N}$ satisfying $N \leq n \leq N + M$.

2.3. The Algebraic and Order Limit Theorems

Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution.

(a) Let $\varepsilon > 0$ be given. Since $x_n \to 0$, there exists an $n \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad |x_n| = x_n < \varepsilon^2 \quad \Leftrightarrow \quad \sqrt{x_n} < \varepsilon.$$

It follows that $\sqrt{x_n} \to 0$.

(b) By the Order Limit Theorem (Theorem 2.3.4) we must have $x \geq 0$. The case $x = 0$ was handled in part (a) so suppose that $x > 0$, which gives $\sqrt{x} > 0$. For each $n \in \mathbf{N}$, observe that

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{\left|\sqrt{x_n} - \sqrt{x}\right|\left(\sqrt{x_n} - \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}.$$

Let $\varepsilon > 0$ be given. Since $x_n \to x$, there exists an $N \in \mathbf{N}$ such that $|x_n - x| < \varepsilon\sqrt{x}$ whenever $n \geq N$. For $n \geq N$ it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \frac{|x_n - x|}{\sqrt{x}} < \varepsilon.$$

Thus $\sqrt{x_n} \to \sqrt{x}$.

Exercise 2.3.2. Using only definition 2.2.3, prove that if $(x_n) \to 2$ then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

Solution.

(a) Let $\varepsilon > 0$ be given. Since $x_n \to 2$, there exists an $N \in \mathbf{N}$ such that $|x_n - 2| < \frac{3\varepsilon}{2}$ whenever $n \geq N$. For such n we then have

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \varepsilon.$$

It follows that $\frac{2x_n - 1}{3} \to 1$.

(b) Since $x_n \to 2$, there is an $N_1 \in \mathbf{N}$ such that $|x_n - 2| < 1$ whenever $n \geq N_1$. For $n \geq N_1$ we then have

$$2 \le |x_n - 2| + |x_n| < 1 + |x_n| \quad \Rightarrow \quad \frac{1}{|x_n|} < 1.$$

Let $\varepsilon > 0$ be given. Since $x_n \to 2$, there is an $N_2 \in \mathbf{N}$ such that $|x_n - 2| < 2\varepsilon$ for $n \ge N_2$. Let $N = \max\{N_1, N_2\}$ and observe that for $n \ge N$ we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2} < \varepsilon.$$

It follows that $\frac{1}{x_n} \to \frac{1}{2}$.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \le y_n \le z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \ge N_1 \quad \Rightarrow \quad |x_n - l| < \varepsilon \quad \Leftrightarrow \quad -\varepsilon < x_n - l < \varepsilon,$$
$$n \ge N_2 \quad \Rightarrow \quad |z_n - l| < \varepsilon \quad \Leftrightarrow \quad -\varepsilon < z_n - l < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Because $x_n - l \le y_n - l \le z_n - l$ for all $n \in \mathbf{N}$, for $n \ge N$ we have

$$-\varepsilon < x_n - l \le y_n - l \le z_n - l < \varepsilon \quad \Rightarrow \quad |y_n - l| < \varepsilon.$$

Thus $\lim_{n\to\infty} y_n = l$.

Exercise 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution. The manipulations of limits in these solutions are justified by the Algebraic Limit Theorem (Theorem 2.3.3).

(a)
$$\lim\left(\frac{1 + 2a_n}{1 + 3a_n - 4a_n^2}\right) = \frac{1 + 2\lim a_n}{1 + 3\lim a_n - 4(\lim a_n)^2} = \frac{1}{1} = 1.$$

(b)
$$\lim\left(\frac{(a_n + 2)^2 - 4}{a_n}\right) = \lim\left(\frac{a_n^2 + 4a_n}{a_n}\right) = \lim(a_n + 4) = \lim a_n + 4 = 4.$$

(c)
$$\lim\left(\frac{\frac{2}{a_n} + 3}{\frac{1}{a_n} + 5}\right) = \lim\left(\frac{2 + 3a_n}{1 + 5a_n}\right) = \frac{2 + 3\lim a_n}{1 + 5\lim a_n} = \frac{2}{1} = 2.$$

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution. (z_n) is the sequence given by

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Suppose that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = L$ for some $L \in \mathbf{R}$ and let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \quad \Rightarrow \quad |x_n - L| < \varepsilon \quad \text{and} \quad n \geq N_2 \quad \Rightarrow \quad |y_n - L| < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$ and suppose $n \in \mathbf{N}$ is such that $n \geq 2N$. If n is odd then $\frac{n+1}{2} \in \mathbf{N}$ and

$$n \geq 2N > 2N - 1 \quad \Rightarrow \quad \frac{n+1}{2} > N \geq N_1 \quad \Rightarrow \quad \left| x_{\frac{n+1}{2}} - L \right| = |z_n - L| < \varepsilon.$$

If n is even then $\frac{n}{2} \in \mathbf{N}$ and

$$n \geq 2N \quad \Rightarrow \quad \frac{n}{2} \geq N \geq N_2 \quad \Rightarrow \quad \left| y_{\frac{n}{2}} - L \right| = |z_n - L| < \varepsilon.$$

Thus $|z_n - L| < \varepsilon$ for any $n \geq N$; it follows that $\lim z_n = L$.

Now suppose that (z_n) is convergent with $\lim z_n = L$ for some $L \in \mathbf{R}$. Let $\varepsilon > 0$ be given. Because $z_n \to L$, there exists an $N \in \mathbf{N}$ such that $|z_n - L| < \varepsilon$ whenever $n \geq N$. For such n we have $2n > 2n - 1 \geq n \geq N$ and thus

$$|x_n - L| = |z_{2n-1} - L| < \varepsilon \quad \text{and} \quad |y_n - L| = |z_{2n} - L| < \varepsilon.$$

It follows that $\lim x_n = \lim y_n = L$.

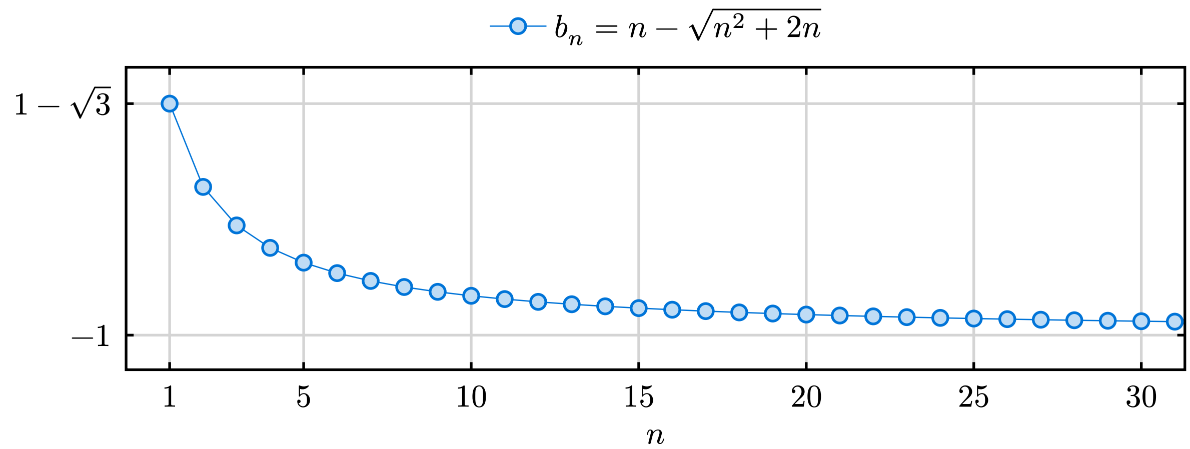
Exercise 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution. Observe that

$$b_n = n - \sqrt{n^2 + 2n} = \frac{\left(n - \sqrt{n^2 + 2n}\right)\left(n + \sqrt{n^2 + 2n}\right)}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}.$$

Thus, using Exercise 2.3.1,

$$\lim b_n = \lim \left(-\frac{2}{1 + \sqrt{1 + \frac{2}{n}}} \right) = \frac{-2}{1 + \sqrt{1 + 2 \lim \frac{1}{n}}} = \frac{-2}{1 + \sqrt{1}} = -1.$$



Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution.

(a) An example is given by $x_n = n$ and $y_n = -n$.

(b) This is impossible. If (x_n) and $(x_n + y_n)$ both converge then by the Algebraic Limit Theorem (Theorem 2.3.3) (y_n) must be convergent and satisfy $\lim y_n = \lim(x_n + y_n) - \lim x_n$.

(c) An example is given by $b_n = \frac{1}{n}$.

(d) This is impossible: $(a_n - b_n)$ must be unbounded. Since (b_n) is convergent, it must be bounded (Theorem 2.3.2), i.e. there is some $B \geq 0$ such that $|b_n| \leq B$ for all $n \in \mathbf{N}$. Let $M \geq 0$ be given. Because (a_n) is unbounded, there exists an $N \in \mathbf{N}$ such that $|a_N| \geq M + B$. Observe that

$$|a_N - b_N| \geq ||a_N| - |b_N|| \geq |a_N| - |b_N| \geq M + B - B = M,$$

where we have used the reverse triangle inequality (Exercise 1.2.6 (d)) for the first inequality. Since M was arbitrary, we see that the sequence $(a_n - b_n)$ is unbounded.

(e) An example is given by $a_n = \frac{1}{n^2}$ and $b_n = n$.

Exercise 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution.

(a) Suppose $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$. The Algebraic Limit Theorem (Theorem 2.3.3) and some simple induction arguments allow us to make the following manipulations:

$$\begin{aligned}\lim p(x_n) &= \lim\left(a_m x_n^m + a_{m-1}x_n^{m-1} + \cdots + a_1 x_n + a_0\right)\\ &= a_m(\lim x_n)^m + a_{m-1}(\lim x_n)^{m-1} + \cdots + a_1 \lim x_n + a_0\\ &= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0\\ &= p(x).\end{aligned}$$

(b) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0,\\ 1 & \text{otherwise,}\end{cases}$$

and the convergent sequence $x_n = \frac{1}{n} \to 0$. We then have $(f(x_n)) = (1, 1, 1, \ldots)$, which converges to $1 \neq 0 = f(0)$.

Exercise 2.3.9.

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution.

(a) There is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Let $\varepsilon > 0$ be given. Because $b_n \to 0$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad |b_n| < \frac{\varepsilon}{M}.$$

Observe that for $n \geq N$ we have

$$|a_n b_n| = |a_n||b_n| \leq M|b_n| < \frac{M\varepsilon}{M} = \varepsilon.$$

It follows that $\lim(a_n b_n) = 0$. We may not use the Algebraic Limit Theorem here since the sequence (a_n) is not necessarily convergent; the hypotheses of that theorem require both sequences (a_n) and (b_n) to be convergent.

(b) If the sequence (a_n) converges to some a then we may use the Algebraic Limit Theorem to conclude that $\lim(a_n b_n) = ab$. If the sequence (a_n) is divergent, then $(a_n b_n)$ must also be divergent. To see this, we will prove the contrapositive, i.e. if $(a_n b_n)$ converges to some $x \in \mathbf{R}$ then (a_n) is convergent. Indeed, since $b \neq 0$, the Algebraic Limit Theorem implies that

$$\lim a_n = \lim\left(\frac{a_n b_n}{b_n}\right) = \frac{x}{b}.$$

(c) Since (b_n) is convergent, it is bounded (Theorem 2.3.2). So we may apply part (a) (with the roles of (a_n) and (b_n) swapped) to conclude that

$$\lim(a_n b_n) = 0 = 0b = ab.$$

Exercise 2.3.10. Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution.

(a) This is false: consider $a_n = b_n = (-1)^n$.

(b) This is true. Let $\varepsilon > 0$ be given. Since $b_n \to b$, there is an $N \in \mathbf{N}$ such that $|b_n - b| < \varepsilon$ whenever $n \geq N$. For such n, the reverse triangle inequality (Exercise 1.2.6 (d)) gives

$$||b_n| - |b|| \leq |b_n - b| < \varepsilon.$$

Thus $\lim|b_n| = |b|$.

(c) This is true. Using the Algebraic Limit Theorem (Theorem 2.3.3), we have

$$\lim b_n = \lim(b_n - a_n + a_n) = \lim(b_n - a_n) + \lim a_n = 0 + a = a.$$

(d) This is true. Since $0 \leq |b_n - b| \leq a_n$ for every $n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.2.3) implies that $\lim|b_n - b| = 0$, i.e. for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad ||b_n - b| - 0| = |b_n - b| < \varepsilon,$$

which is exactly the statement $\lim_{n\to\infty} b_n = b$.

Exercise 2.3.11 (Cesaro Means).

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution.

(a) Suppose $\lim x_n = x$ and let $\varepsilon > 0$ be given. Since $x_n \to x$, there is a positive integer $N_1 \in \mathbf{N}$ such that

$$n \geq N_1 \quad \Rightarrow \quad |x_n - x| < \frac{\varepsilon}{2}.$$

Given this N_1, notice that the sequence

$$\left(\frac{\sum_{k=1}^{N_1} |x_k - x|}{n} \right)$$

has non-negative terms and converges to zero as $n \to \infty$ (the numerator is a constant). It follows that there is an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \quad \Rightarrow \quad \frac{\sum_{k=1}^{N_1} |x_k - x|}{n} < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $n \geq N + 1$ we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{\sum_{k=1}^{n} x_k}{n} - \frac{nx}{n} \right| \\
&= \left| \frac{\sum_{k=1}^{n} (x_k - x)}{n} \right| \\
&\leq \frac{\sum_{k=1}^{N_1} |x_k - x|}{n} + \frac{\sum_{k=N_1+1}^{n} |x_k - x|}{n} \\
&< \frac{\varepsilon}{2} + \left(\frac{n - N_1}{n} \right) \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus $\lim_{n\to\infty} y_n = x$.

(b) Consider the divergent sequence $x_n = (-1)^{n+1}$. The sequence of averages (y_n) is then

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which satisfies $\lim y_n = 0$.

Exercise 2.3.12. A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every (a_n) is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution.

(a) This is true. For any $b \in B$ we have $b \leq a_n$ for all $n \in \mathbf{N}$; the Order Limit Theorem (Theorem 2.3.4) then implies that $b \leq a$ and it follows that a is an upper bound of B.

(b) This is true. Observe that for a real number x we have

$$x \notin (0, 1) \quad \Leftrightarrow \quad x \leq 0 \text{ or } x \geq 1 \quad \Leftrightarrow \quad \left| x - \frac{1}{2} \right| \geq \frac{1}{2}.$$

So for each $n \in \mathbf{N}$ we have $\left|a_n - \frac{1}{2}\right| \geq \frac{1}{2}$. The Algebraic Limit Theorem (Theorem 2.3.3) and Exercise 2.3.10 (b) imply that $\lim \left|a_n - \frac{1}{2}\right| = \left|a - \frac{1}{2}\right|$, and thus the Order Limit Theorem (Theorem 2.3.4) gives us $\left|a - \frac{1}{2}\right| \geq \frac{1}{2}$. It follows that a belongs to the complement of $(0, 1)$.

(c) This is false. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), for each $n \in \mathbf{N}$ we may pick a rational number a_n satisfying $\sqrt{2} < a_n < \sqrt{2} + \frac{1}{n}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim a_n = \sqrt{2}$, which is an irrational number.

Exercise 2.3.13 (Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \varepsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) = b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution.

(a) We apply the Algebraic Limit Theorem (Theorem 2.3.3):

$$\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty}\left(\frac{m}{m+n}\right) = \lim_{m\to\infty}\left(\frac{1}{1+\frac{n}{m}}\right) = \frac{1}{1+n\lim_{m\to\infty}\frac{1}{m}} = \frac{1}{1} = 1.$$

Thus $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = 1$. Similarly,

$$\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty}\left(\frac{m}{m+n}\right) = \lim_{n\to\infty}\left(\frac{\frac{m}{n}}{1+\frac{m}{n}}\right) = \frac{m\lim_{n\to\infty}\frac{1}{n}}{1+n\lim_{m\to\infty}\frac{1}{n}} = \frac{0}{1} = 0.$$

Thus $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

(b) For $a_{mn} = \frac{1}{m+n}$, we claim that $\lim_{m,n\to\infty} a_{mn} = 0$. Indeed, let $\varepsilon > 0$ be given and let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \varepsilon$. For any $m, n \geq N$ it follows that

$$|a_{mn}| = \frac{1}{m+n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus $\lim_{m,n\to\infty} a_{mn} = 0$. The two iterated limits also exist and are equal to 0. Because $0 < \frac{1}{m+n} < \frac{1}{m}$ for all $m, n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.3.3) implies that $\lim_{m\to\infty} a_{mn} = 0$ and it follows that $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = 0$; a similar argument shows that $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

Now let $a_{mn} = \frac{mn}{m^2+n^2}$. We will show that $\lim_{m,n\to\infty} a_{mn}$ does not exist by using the following lemma.

Lemma L.6. Let $a_{m,n}$ be a doubly indexed array and suppose that $\lim_{m,n\to\infty} a_{m,n} = L$ for some $L \in \mathbf{R}$. If $\theta : \mathbf{N} \to \mathbf{N}$ satisfies $\lim_{m\to\infty} \theta(m) = \infty$, then $\lim_{m\to\infty} a_{m,\theta(m)} = L$.

Proof. Let $\varepsilon > 0$ be given. Because $\lim_{m,n\to\infty} a_{m,n} = L$, there is an $M_1 \in \mathbf{N}$ such that

$$m, n \geq M_1 \quad \Rightarrow \quad \left|a_{m,n} - L\right| < \varepsilon,$$

and because $\lim_{m\to\infty} \theta(m) = \infty$ there is an $M_2 \in \mathbf{N}$ such that $\theta(m) \geq M_1$ whenever $m \geq M_2$. Let $M = \max\{M_1, M_2\}$ and suppose that $m \geq M$. It follows that $m \geq M_1$ and that $\theta(m) \geq M_1$ and thus $\left|a_{m,\theta(m)} - L\right| < \varepsilon$. □

An immediate corollary of Lemma L.6 is that if $\lim_{m,n\to\infty} a_{m,n}$ exists, then

$$\lim_{m\to\infty} a_{m,\theta_1(m)} = \lim_{m\to\infty} a_{m,\theta_2(m)} = \lim_{m,n\to\infty} a_{m,n}$$

for any functions $\theta_1, \theta_2 : \mathbf{N} \to \mathbf{N}$ satisfying $\lim_{m\to\infty} \theta_i(m) = \infty$. Now observe that

$$a_{m,m} = \frac{m^2}{m^2 + m^2} = \frac{1}{2} \quad \Rightarrow \quad \lim_{m\to\infty} a_{m,m} = \frac{1}{2},$$

$$a_{m,2m} = \frac{2m^2}{m^2 + 4m^2} = \frac{2}{5} \quad \Rightarrow \quad \lim_{m\to\infty} a_{m,2m} = \frac{2}{5}.$$

It follows from the contrapositive of the corollary above (we are taking $\theta_1(m) = m$ and $\theta_2(m) = 2m$) that $\lim_{m,n\to\infty} a_{mn}$ does not exist. However, the two iterated limits do exist and are equal to 0. Using the Algebraic Limit Theorem (Theorem 2.3.3), for any $n \in \mathbf{N}$ we have

$$\lim_{m\to\infty}\left(\frac{mn}{m^2+n^2}\right) = \lim_{m\to\infty}\left(\frac{\frac{n}{m}}{1+\frac{n^2}{m^2}}\right) = \frac{n \lim_{m\to\infty} \frac{1}{m}}{1 + n^2 \lim_{m\to\infty} \frac{1}{m^2}} = \frac{0}{1} = 0.$$

It follows that $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = 0$ and a similar argument shows that $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

(c) Let $a_{mn} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$. We claim that $\lim_{m,n\to\infty} a_{mn} = 0$. Let $\varepsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. For $m, n \geq N$ we then have

$$|a_{mn}| = \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)\right| = \frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_{m,n\to\infty} a_{mn} = 0$. However, neither iterated limit exists. Fix $n \in \mathbf{N}$ and observe that

$$\begin{aligned}
|a_{m,n} - a_{m+1,n}| &= \left|(-1)^{m+n}\left(\frac{1}{m}+\frac{1}{n}\right) - (-1)^{m+n+1}\left(\frac{1}{m+1}+\frac{1}{n}\right)\right| \\
&= \left|(-1)^{m+n}\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{m+1}+\frac{1}{n}\right)\right| \\
&= \frac{1}{m}+\frac{1}{m+1}+\frac{2}{n} \\
&\geq \frac{2}{n}.
\end{aligned}$$

Because $n \in \mathbf{N}$ is fixed, this implies that the sequence $(a_{m,n} - a_{m+1,n})_{m=1}^{\infty}$ cannot converge to 0. Now, for a fixed $n \in \mathbf{N}$, the Algebraic Limit Theorem (Theorem 2.3.3) gives us

$$\lim_{m\to\infty} a_{m,n} \text{ exists } \quad\Rightarrow\quad \lim_{m\to\infty}(a_{m,n} - a_{m+1,n}) = 0.$$

Thus, given that $(a_{m,n} - a_{m+1,n})_{m=1}^{\infty}$ does not converge to zero, it must be the case that $\lim_{m\to\infty} a_{m,n}$ does not exist. Since this is true for any $n \in \mathbf{N}$, we see that the iterated limit $\lim_{n\to\infty}(\lim_{m\to\infty} a_{m,n})$ does not exist. Using the symmetry of $a_{m,n}$ and swapping the roles of m and n in our argument shows that the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{m,n})$ does not exist either.

(d) First, using our hypothesis that $\lim_{n\to\infty} a_{m,n} = b_m$ for each fixed $m \in \mathbf{N}$, the Algebraic Limit Theorem (Theorem 2.3.3), and Exercise 2.3.10 (b), notice that $\lim_{n\to\infty}|a_{m,n} - a| = |b_m - a|$ for any $m \in \mathbf{N}$.

Now let $\varepsilon > 0$ be given. Because $\lim_{m,n\to\infty} a_{m,n} = a$, there is an $N \in \mathbf{N}$ such that $|a_{m,n} - a| < \frac{\varepsilon}{2}$ whenever $m, n \geq N$. Suppose that $m \geq N$ and observe that, by the Order Limit Theorem (Theorem 2.3.4),

$$|a_{m,n} - a| < \frac{\varepsilon}{2} \text{ for all } n \geq N \quad\Rightarrow\quad \lim_{n\to\infty}|a_{m,n} - a| = |b_m - a| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus $|b_m - a| < \varepsilon$ whenever $m \geq N$ and it follows that $\lim_{m\to\infty} b_m = a$.

(e) If the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ exists, then it must be the case that for each fixed $m \in \mathbf{N}$, the limit $\lim_{n\to\infty} a_{mn}$ exists. Part (d) then implies that

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

Swapping the roles of m and n and repeating the above argument shows that

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

2.4. The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1.

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $x_{n+1} \leq x_n$ and $x_n \geq -1$. We will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = 3$ and $x_2 = 1$, we see that $P(1)$ holds. Suppose that $P(1), ..., P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \leq x_n \leq \cdots \leq x_1 = 3 \quad \Rightarrow \quad 1 \leq 4 - x_n \leq 4 - x_{n+1} \quad \Rightarrow \quad \frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_n},$$

i.e. $x_{n+2} \leq x_{n+1}$. Furthermore,

$$-1 \leq x_n \leq 3 \quad \Rightarrow \quad 1 \leq 4 - x_n \leq 5 \quad \Rightarrow \quad x_{n+1} = \frac{1}{4 - x_n} \geq \frac{1}{5} > -1.$$

Thus $P(n+1)$ holds. This completes the induction step.

We have now shown that the sequence (x_n) is bounded below and decreasing. The Monotone Convergence Theorem (Theorem 2.4.2) allows us to conclude that the sequence converges.

(b) If (x_n) is any convergent sequence with $\lim x_n = x$, then the sequence (y_n) given by $y_n = x_{n+k}$ for any $k \in \mathbf{N}$ is also convergent with $\lim y_n = x$. To see this, let $\varepsilon > 0$ be given. Since $\lim x_n = x$, there exists an $N \in \mathbf{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq N$. Suppose $n \geq \max\{N - k, 1\}$, so that $n + k \geq N$, and observe that

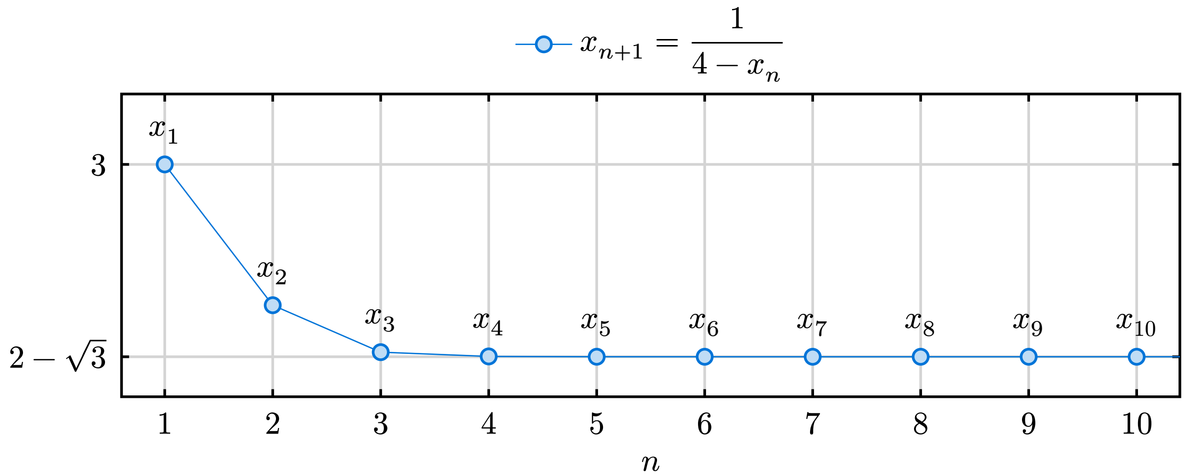
$$|y_n - x| = \left|x_{n+k} - x\right| < \varepsilon.$$

Thus $\lim y_n = x$.

(c) By parts (a) and (b) we have $\lim x_n = \lim x_{n+1} = x$ for some $x \in \mathbf{R}$. Taking the limit on both sides of the recursive equation and using the Algebraic Limit Theorem (Theorem 2.3.3), we find that

$$\lim x_{n+1} = \frac{1}{4 - \lim x_n} \quad \Leftrightarrow \quad x = \frac{1}{4-x} \quad \Leftrightarrow \quad x^2 - 4x + 1 = 0.$$

This quadratic equation has solutions $x = 2 \pm \sqrt{3}$. Since (x_n) is decreasing and $x_2 = 1$, the Order Limit Theorem (Theorem 2.3.4) implies that $\lim x_n = x \leq 1 < 2 + \sqrt{3}$ and so we may discard the solution $x = 2 + \sqrt{3}$ to conclude that $\lim x_n = 2 - \sqrt{3}$.



Exercise 2.4.2.

(a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution.

(a) The problem is we have assumed that $\lim y_n$ exists. Looking at the first few terms of the sequence $y_1 = 1, y_2 = 2, y_3 = 1, y_4 = 2, \ldots$, we see that in fact the sequence oscillates and does not converge.

(b) The strategy works this time. Let $P(n)$ be the statement that $y_{n+1} \geq y_n$ and $y_n \leq 3$. We will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $y_1 = 1$ and $y_2 = 2$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} \geq y_n \geq \cdots \geq y_1 = 1 \quad \Rightarrow \quad \frac{1}{y_{n+1}} \leq \frac{1}{y_n} \quad \Rightarrow \quad 3 - \frac{1}{y_{n+1}} \geq 3 - \frac{1}{y_n},$$

i.e. $y_{n+2} \geq y_{n+1}$. Furthermore,

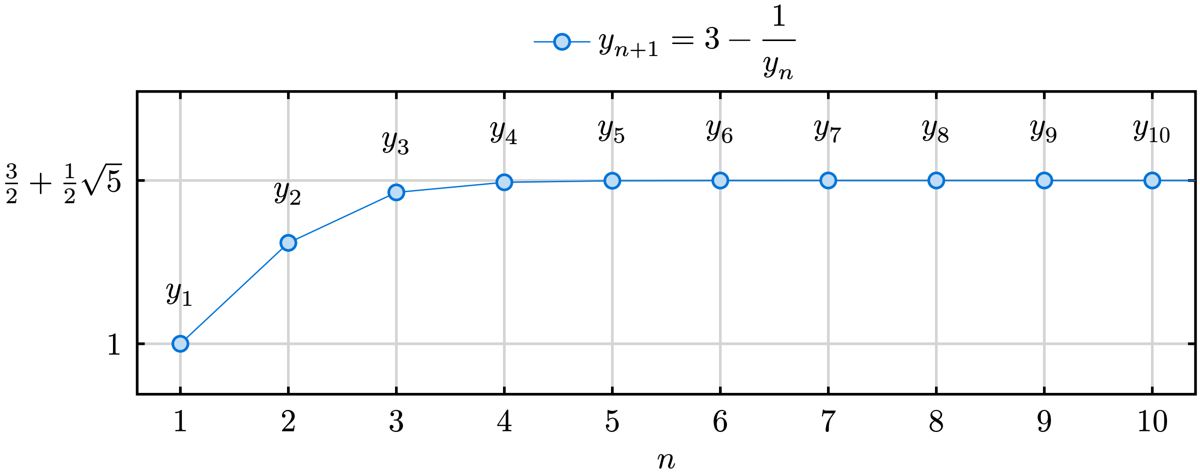
$$1 \le y_n \le 3 \quad \Rightarrow \quad \frac{1}{3} \le \frac{1}{y_n} \quad \Rightarrow \quad y_{n+1} = 3 - \frac{1}{y_n} \le \frac{8}{3} < 3.$$

Thus $P(n+1)$ holds. This completes the induction step.

We have now shown that (y_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim y_n = y$ for some $y \in \mathbf{R}$. Given this, we can take the limit across the recursive equation to obtain:

$$\lim y_{n+1} = 3 - \frac{1}{\lim y_n} \quad \Leftrightarrow \quad y = 3 - \frac{1}{y} \quad \Leftrightarrow \quad y^2 - 3y + 1 = 0.$$

This quadratic equation has solutions $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$. Since (y_n) is increasing and $y_2 = 2$, we must have $y \ge 2 > \frac{3}{2} - \frac{1}{2}\sqrt{5}$ and so we may discard the solution $y = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ to conclude that $\lim y_n = \frac{3}{2} + \frac{1}{2}\sqrt{5}$.



Exercise 2.4.3.

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Solution.

(a) Let $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2 + x_n}$, and let $P(n)$ be the statement that $x_{n+1} \ge x_n$ and $x_n \le 2$. We will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2+\sqrt{2}}$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \geq x_n \geq \cdots \geq x_1 = \sqrt{2} \quad \Rightarrow \quad \sqrt{2 + x_{n+1}} \geq \sqrt{2 + x_n},$$

i.e. $x_{n+2} \geq x_{n+1}$. Furthermore,

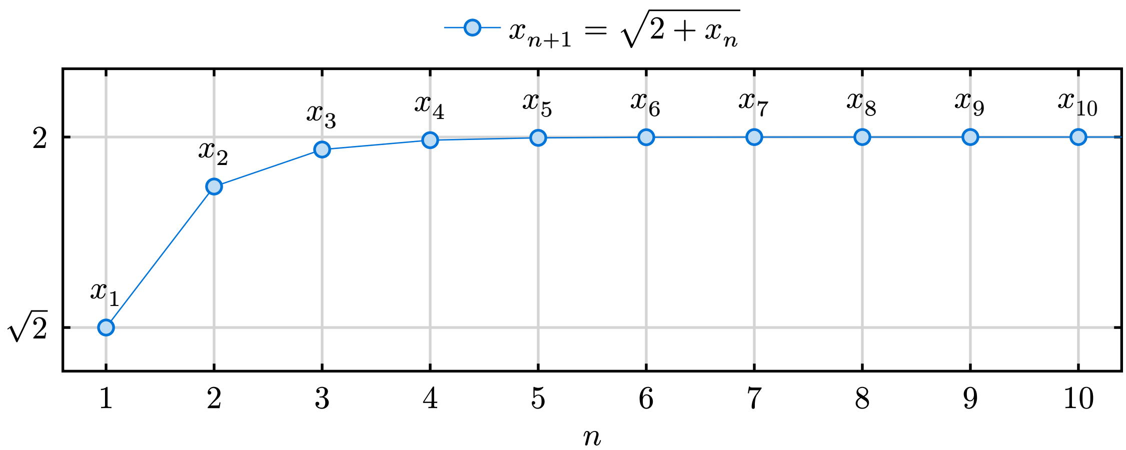
$$\sqrt{2} \leq x_n \leq 2 \quad \Rightarrow \quad \sqrt{2 + x_n} \leq \sqrt{4} = 2.$$

Thus $P(n+1)$ holds. This completes the induction step.

We have now shown that the sequence (x_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim x_n = x$ for some $x \in \mathbf{R}$. We may now take the limit on both sides of the recursive equation and use Exercise 2.3.1 to see that

$$\lim x_{n+1} = \sqrt{2 + \lim x_n} \quad \Rightarrow \quad x = \sqrt{2 + x} \quad \Rightarrow \quad x^2 - x - 2 = (x-2)(x+1) = 0.$$

So $x = 2$ or $x = -1$. Since the sequence is increasing and $x_1 = \sqrt{2}$, we must have $x \geq \sqrt{2} > -1$ and thus $\lim x_n = 2$.



(b) The sequence does converge. Let $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2x_n}$, and let $P(n)$ be the statement that $x_{n+1} \geq x_n$ and $x_n \leq 2$. We will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2\sqrt{2}}$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \geq x_n \geq \cdots \geq x_1 = \sqrt{2} \quad \Rightarrow \quad \sqrt{2x_{n+1}} \geq \sqrt{2x_n},$$

i.e. $x_{n+2} \geq x_{n+1}$. Furthermore,

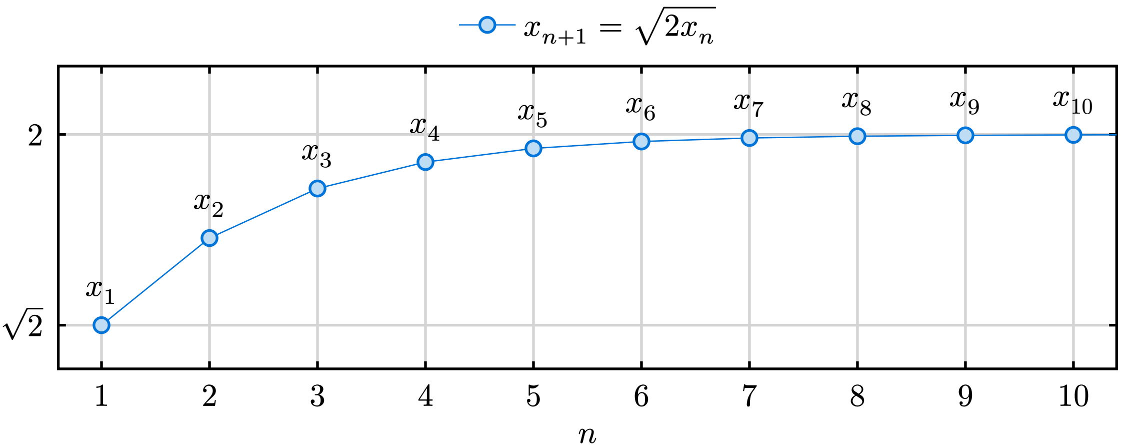
$$\sqrt{2} \leq x_n \leq 2 \quad \Rightarrow \quad \sqrt{2x_n} \leq \sqrt{4} = 2.$$

Thus $P(n+1)$ holds. This completes the induction step.

We have now shown that the sequence (x_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim x_n = x$ for some $x \in \mathbf{R}$. We may now take the limit on both sides of the recursive equation and use Exercise 2.3.1 to see that

$$\lim x_{n+1} = \sqrt{2 \lim x_n} \quad \Rightarrow \quad x = \sqrt{2x} \quad \Rightarrow \quad x^2 - 2x = x(x-2) = 0.$$

Thus $x = 2$ or $x = 0$. Since the sequence is increasing and $x_1 = \sqrt{2}$, we must have $x \geq \sqrt{2} > 0$ and so $\lim x_n = 2$.



Exercise 2.4.4.

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution.

(a) Assuming that any bounded monotone sequence converges, we want to prove part (i) of Theorem 1.4.2: for any $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ satisfying $n > x$. Part (ii) of Theorem 1.4.2 will then follow by taking $x = \frac{1}{y}$ in part (i). Let $x \in \mathbf{R}$ be given and, seeking a contradiction, suppose that $n \leq x$ for each $n \in \mathbf{N}$. It follows that the increasing sequence $(1, 2, 3, \ldots)$ is bounded above and hence by assumption converges to some $y \in \mathbf{R}$. There then exists an $N \in \mathbf{N}$ such that $|n - y| < \frac{1}{2}$ whenever $n \geq N$. However, this implies that

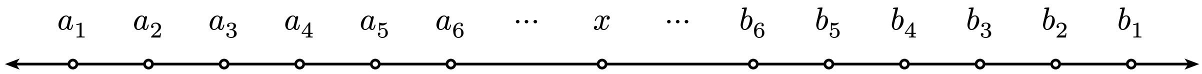
$$1 = |N + 1 - y + y - N| \leq |N + 1 - y| + |N - y| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e. $1 < 1$, a contradiction. We may conclude that there exists some $n \in \mathbf{N}$ such that $n > x$.

(b) Assuming that any bounded monotone sequence converges, we want to prove that any sequence of nested intervals $I_n = [a_n, b_n]$ has a non-empty intersection $\bigcap_{n=1}^{\infty} I_n$. Consider the sequence (a_n) of left-hand endpoints, which must be increasing because the intervals are nested. Moreover, this sequence is bounded above by any right-hand endpoint. Thus, by assumption, this sequence converges to some $x \in \mathbf{R}$. Notice that for any $n \in \mathbf{N}$ we have $a_n \leq a_m \leq b_m \leq b_n$ for all $m \geq n$. The Order Limit Theorem (Theorem 2.3.4) then implies that

$$x = \lim_{m \to \infty} a_m \leq b_n \quad \text{and} \quad a_n \leq \lim_{m \to \infty} a_m = x.$$

It follows that $a_n \leq x \leq b_n$ for all $n \in \mathbf{N}$, i.e. $x \in \bigcap_{n=1}^{\infty} I_n$.



(In the general case the endpoints will not be so evenly spaced, although the ordering will be the same.)

Exercise 2.4.5 (Calculating Square Roots). Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution.

(a) Let $P(n)$ be the statement that $x_n \geq \sqrt{2}$. We will use induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. The truth of $P(1)$ is clear, so suppose that $P(n)$ holds for some $n \in \mathbf{N}$. Observe that

$$\left(x_n - \sqrt{2}\right)^2 = x_n^2 - 2\sqrt{2}x_n + 2 \geq 0.$$

Our induction hypothesis guarantees that $x_n \geq \sqrt{2} > 0$ and so we may divide by x_n to obtain the inequality

$$x_n - 2\sqrt{2} + \frac{2}{x_n} \geq 0 \quad \Leftrightarrow \quad \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{2},$$

i.e. $x_{n+1} \geq \sqrt{2}$. This completes the induction step and thus, in particular, $x_n^2 \geq 2$ for each $n \in \mathbf{N}$. For any $n \in \mathbf{N}$ we then have

$$x_n^2 - 2 \geq 0 \quad \Leftrightarrow \quad \frac{x_n}{2} - \frac{1}{x_n} \geq 0 \quad \Leftrightarrow \quad x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq 0 \quad \Leftrightarrow \quad x_n - x_{n+1} \geq 0.$$

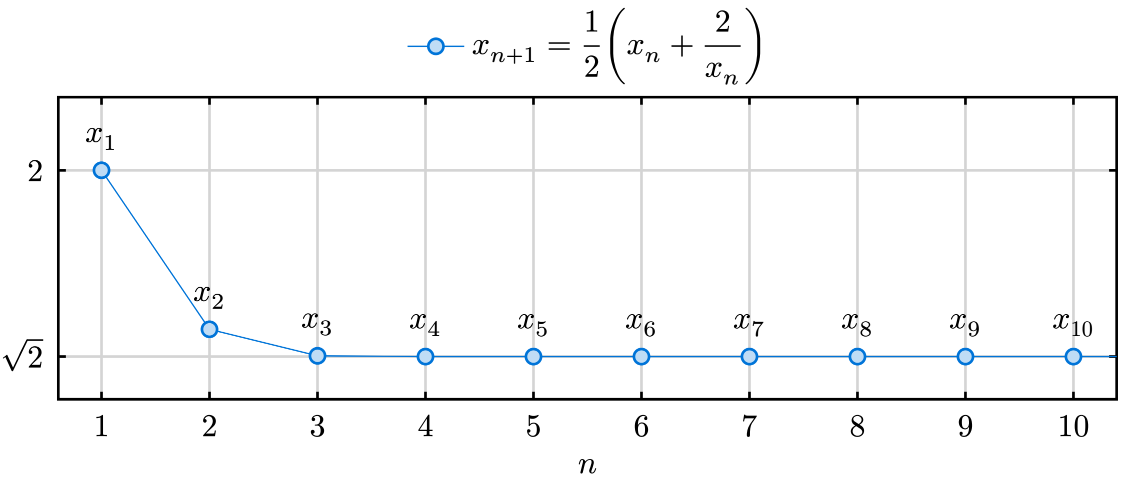
Thus $x_{n+1} \leq x_n$ for all $n \in \mathbf{N}$.

We have now shown that the sequence (x_n) is decreasing and bounded below. The Monotone Convergence Theorem (Theorem 2.4.2) then implies that $\lim x_n = x$ for some

$x \in \mathbf{R}$, which must satisfy $x \geq \sqrt{2} > 0$ by the Order Limit Theorem (Theorem 2.3.4). We can now take the limit across the recursive equation:

$$\lim x_{n+1} = \frac{1}{2}\left(\lim x_n + \frac{2}{\lim x_n}\right) \quad \Leftrightarrow \quad x = \frac{1}{2}\left(x + \frac{2}{x}\right) \quad \Leftrightarrow \quad x^2 = 2.$$

Since $x \geq \sqrt{2}$ we may conclude that $x = \sqrt{2}$.



(b) For $c \geq 0$, let $x_1 = 1 + c$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

Repeating the argument given in part (a), replacing 2 with c where appropriate, shows that $\lim x_n = \sqrt{c}$. For the base case of the induction argument, note that

$$x_1 = 1 + c \geq 1 \quad \Rightarrow \quad x_1 \geq \sqrt{1+c} > \sqrt{c}.$$

Exercise 2.4.6 (Arithmetic-Geometric Mean).

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution.

(a) Observe that

$$0 \leq (x-y)^2 \quad \Leftrightarrow \quad 0 \leq x^2 - 2xy + y^2 \quad \Leftrightarrow \quad 4xy \leq x^2 + 2xy + y^2 \quad \Leftrightarrow \quad 4xy \leq (x+y)^2.$$

Since x and y are both positive, this implies that $\sqrt{xy} \leq \frac{x+y}{2}$.

(b) By part (a) we have $x_n \leq y_n$ for all $n \in \mathbf{N}$. It follows that

$$y_{n+1} = \frac{x_n + y_n}{2} \le \frac{y_n + y_n}{2} = y_n \quad \text{and} \quad x_{n+1} = \sqrt{x_n y_n} \ge \sqrt{x_n^2} = x_n.$$

Thus (x_n) is increasing and (y_n) is decreasing. Furthermore, (y_n) is bounded below: for any $n \in \mathbf{N}$, we have $y_n \ge x_n \ge \cdots \ge x_1$. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that $\lim y_n = y$ for some $y \in \mathbf{R}$. The Algebraic Limit Theorem (Theorem 2.3.3) then gives

$$x_n = 2y_{n+1} - y_n \quad \Rightarrow \quad \lim x_n = 2 \lim y_{n+1} - \lim y_n = 2y - y = y.$$

Exercise 2.4.7 (Limit Superior). Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \ge n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \le \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution.

(a) Suppose $M > 0$ is the bound for (a_n), i.e. $|a_n| \le M$ for all $n \in \mathbf{N}$. It follows that $y_n \ge a_n \ge -M$ for $n \in \mathbf{N}$, so that the sequence (y_n) is bounded below. Furthermore, for any $n \in \mathbf{N}$ we have

$$\sup\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\} \le \sup\{a_n, a_{n+1}, a_{n+2}, a_{n+3}, \ldots\},$$

i.e. $y_{n+1} \le y_n$. Thus the sequence (y_n) is decreasing and bounded below and hence converges by the Monotone Convergence Theorem (Theorem 2.4.2).

(b) Let $x_n = \inf\{a_k : k \ge n\}$. As in part (a), we can show that this sequence is bounded above, increasing, and hence convergent. We then define the limit inferior as $\liminf a_n = \lim x_n$.

(c) The infimum of a bounded set is always less than or equal to the supremum of that set, so we have $x_n \le y_n$ for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) then implies that $\lim x_n \le \lim y_n$, i.e. $\liminf a_n \le \limsup a_n$.

For an example of a bounded sequence where this inequality is strict, consider the sequence $a_n = (-1)^n$. For this sequence we have $(x_n) = (-1, -1, -1, \ldots)$ and $(y_n) = (1, 1, 1, \ldots)$, so that $\liminf a_n = -1 < 1 = \limsup a_n$.

(d) Suppose $\liminf a_n = \limsup a_n$. Since $x_n \leq a_n \leq y_n$ for all $n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.3.3) implies that (a_n) converges and that $\liminf a_n = \limsup a_n = \lim a_n$.

Now suppose that $\lim a_n = a$ for some $a \in \mathbf{R}$ and let $\varepsilon > 0$ be given. Since $a_n \to a$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}.$$

This implies that $a - \frac{\varepsilon}{2}$ is a lower bound for $\{a_k : k \geq N\}$ and that $a + \frac{\varepsilon}{2}$ is an upper bound for $\{a_k : k \geq N\}$. It follows that $a - \frac{\varepsilon}{2} \leq x_N \leq a_N \leq y_N \leq \frac{\varepsilon}{2}$ and hence, since (x_n) is increasing and (y_n) is decreasing,

$$n \geq N \quad \Rightarrow \quad a - \varepsilon < x_N \leq x_n \leq a_n \leq y_n \leq y_N < a + \varepsilon.$$

Thus $|x_n - a| < \varepsilon$ and $|y_n - a| < \varepsilon$ for all $n \geq N$. We may conclude that

$$\liminf a_n = \limsup a_n = \lim a_n = a.$$

Exercise 2.4.8. For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad \text{(c)} \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

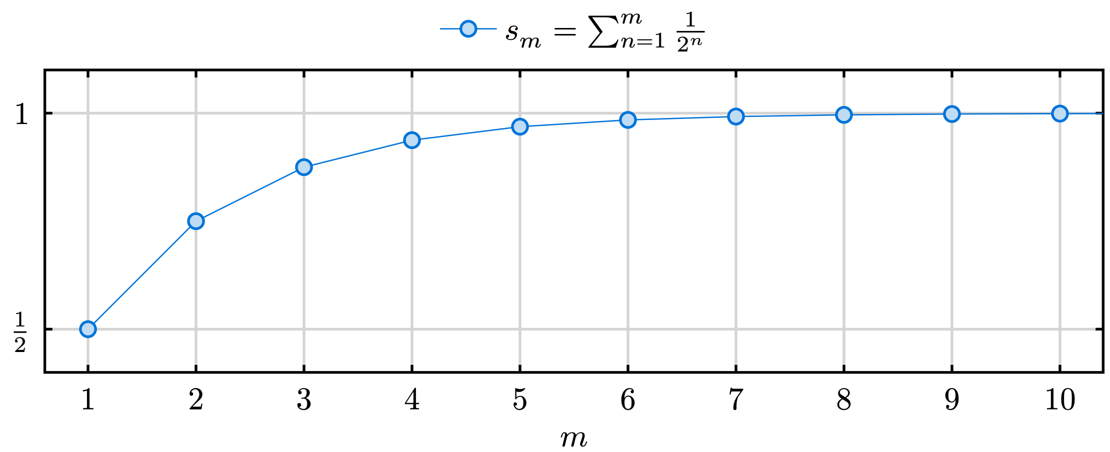
(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution. For each series, let (s_m) be its sequence of partial sums.

(a) Here we have

$$s_m = \frac{1}{2} + \cdots + \frac{1}{2^m} \quad \Rightarrow \quad 2s_m = 1 + \cdots + \frac{1}{2^{m-1}}$$
$$\Rightarrow \quad 2s_m = \frac{1 - 2^{-m}}{1 - \frac{1}{2}} \quad \Rightarrow \quad s_m = 1 - \frac{1}{2^m},$$

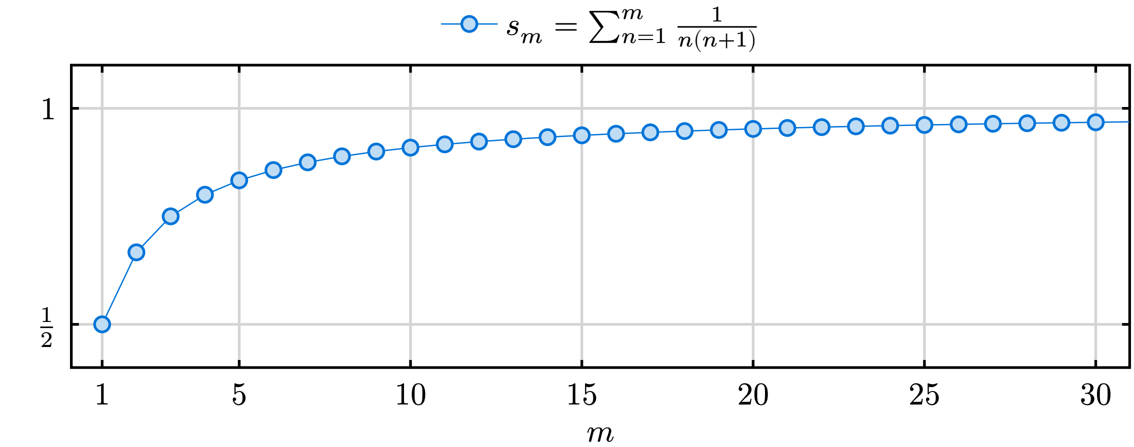
where we have used the formula $(1 - x)(1 + x + \cdots + x^n) = 1 - x^{n+1}$. It follows that $\lim s_m = 1$.



(b) For this series,

$$
\begin{aligned}
s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} &= \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m-1}\right) = 1 - \frac{1}{m+1}.
\end{aligned}
$$

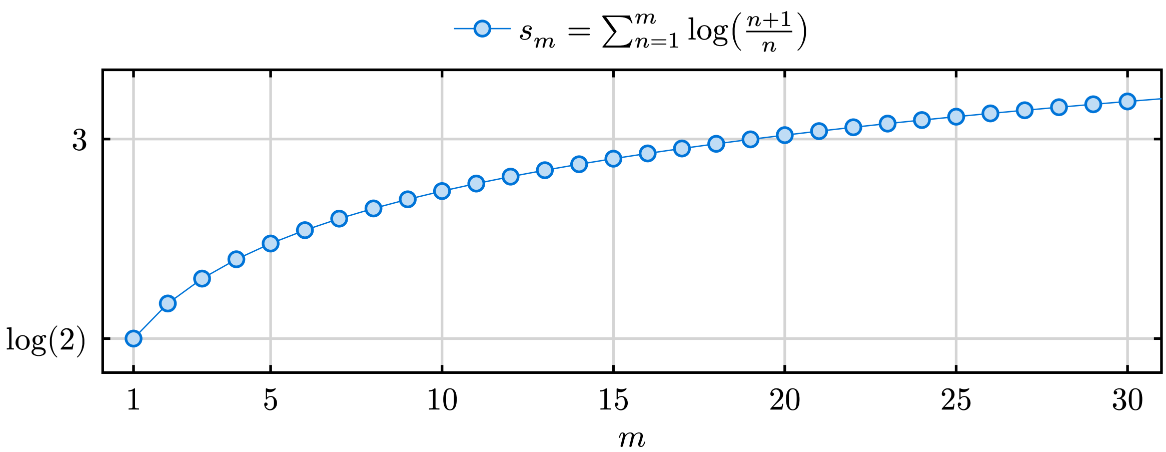
It follows that $\lim s_m = 1$.



(c) We have

$$
\begin{aligned}
s_m &= \sum_{n=1}^{m} \log\left(\frac{n+1}{n}\right) \\
&= \sum_{n=1}^{m} (\log(n+1) - \log(n)) \\
&= (\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(m+1) - \log(m)) \\
&= \log(m+1),
\end{aligned}
$$

which is unbounded and hence not convergent.



Exercise 2.4.9. Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution. Define the sequences of partial sums

$$s_m = b_1 + b_2 + \cdots + b_m \quad \text{and} \quad t_m = b_1 + 2b_2 + \cdots + 2^m b_{2^m}.$$

We will use induction to show that $t_m \leq 2s_{2^m}$ for each $m \in \mathbf{N}$. For the base case $m = 1$ we have

$$t_1 = b_1 + 2b_2 \leq 2b_1 + 2b_2 = 2s_2,$$

where we have used that b_1 is non-negative. Suppose that the inequality holds for some $m \in \mathbf{N}$. If $j \in \{1, ..., 2^m\}$, then $2^m + j \leq 2^{m+1}$; because the sequence (b_n) is decreasing, we then have $b_{2^{m+1}} \leq b_{2^m+j}$. Summing this inequality over all such j gives us $2^m b_{2^{m+1}} \leq \sum_{j=1}^{2^m} b_{2^m+j}$, and combining this with our induction hypothesis we obtain

$$t_{m+1} = t^m + 2^{m+1} b_{2^{m+1}} \leq 2s_{2^m} + 2\sum_{j=1}^{2^m} b_{2^m+j} = 2s_{2^{m+1}}.$$

This completes the induction step.

Since each b_n is non-negative, both sequences of partial sums (s_m) and (t_m) are increasing. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that the convergence of each series is equivalent to the boundedness of the respective sequence of partial sums. Given this, we want to show that if (t_m) is unbounded then so is (s_m); this follows immediately from the inequality $t_m \leq 2s_{2^m}$.

Exercise 2.4.10 (Infinite Products). A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution.

(a) For $a_n = \frac{1}{n}$, observe that

$$p_m = \prod_{n=1}^{m}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{m}\left(\frac{n+1}{n}\right) = 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{m}{m-1}\cdot\frac{m+1}{m}$$
$$= \frac{2}{2}\cdot\frac{3}{3}\cdot\frac{4}{4}\cdots\frac{m}{m}\cdot(m+1) = m+1.$$

It follows that (p_m) does not converge.

For $a_n = \frac{1}{n^2}$, the first few partial products are

$$p_1 = 2, \qquad p_4 = 2.951,$$
$$p_2 = 2.5, \qquad p_5 = 3.069,$$
$$p_3 \approx 2.778, \qquad p_6 \approx 3.155.$$

It looks like the partial products could be bounded. We conjecture that this infinite product converges. Indeed, part (b) proves our conjecture, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent series.

(b) Let

$$s_m = \sum_{n=1}^{m} a_n \quad \text{and} \quad p_m = \prod_{n=1}^{m}(1+a_n).$$

Because $a_n \geq 0$ for all $n \in \mathbf{N}$, the sequence of partial sums and the sequence of partial products are both non-negative and increasing. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that the convergence of each sequence is equivalent to the boundedness of that sequence. By multiplying out the terms in the partial product p_m, we would obtain the sum s_m and some other non-negative terms; it follows that $s_m \leq p_m$. The hint gives us

$$p_m = \prod_{n=1}^{m}(1+a_n) \leq \prod_{n=1}^{m} 3^{a_n} = 3^{\sum_{n=1}^{m} a_n} = 3^{s_m}.$$

So we have the inequalities $s_m \leq p_m \leq 3^{s_m}$. It follows that any bound of (p_m) is also a bound of (s_m), and that if $M > 0$ is a bound of (s_m) then 3^M is a bound of (p_m). Thus $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

2.5. Subsequences and the Bolzano-Weierstrass Theorem

Exercise 2.5.1. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set

$$\{1, 1/2, 1/3, 1/4, 1/5, ...\}.$$

(d) A sequence that contains subsequences converging to every point in the infinite set

$$\{1, 1/2, 1/3, 1/4, 1/5, ...\},$$

but no subsequences converging to points outside of this set.

Solution.

(a) This is impossible. If a sequence (a_n) has a bounded subsequence (a_{n_k}), then by the Bolzano-Weierstrass Theorem (Theorem 2.5.5) there must be a convergent subsequence $(a_{n_{k_\ell}})$, which is also a convergent subsequence of the original sequence (a_n).

(b) Consider the sequence

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}, ...\right),$$

i.e. the sequence (a_n) given by

$$a_n = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd,} \\ 1 - \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

This sequence does not contain 0 or 1 as a term, the subsequence (a_{2n-1}) converges to 0, and the subsequence (a_{2n}) converges to 1.

(c) Consider the following infinite array:

$$
\begin{array}{cccccccccccc}
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots \\
& \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots \\
& \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots \\
& \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots \\
& \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots \\
& & & & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

Let (a_n) be the sequence obtained by following the diagonals of this array, i.e.

$$(a_n) = \left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \dots\right).$$

(d) This is impossible. Suppose that (a_n) is a sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}.$$

We will construct a subsequence of (a_n) converging to 0. Since there is a subsequence converging to 1, there must be some index n_1 such that

$$\left|a_{n_1} - 1\right| < 1 \quad \Leftrightarrow \quad 0 < a_{n_1} < 2.$$

Since there is a subsequence converging to $\frac{1}{2}$, there must be some index $n_2 > n_1$ such that

$$\left|a_{n_2} - \frac{1}{2}\right| < \frac{1}{2} \quad \Leftrightarrow \quad 0 < a_{n_2} < 1.$$

We continue in this manner, obtaining a subsequence (a_{n_k}) satisfying $0 < a_{n_k} < \frac{2}{k}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim_{k\to\infty} a_{n_k} = 0$.

Exercise 2.5.2. Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution.

(a) This is true. By assumption the subsequence $(x_2, x_3, x_4, \ldots)$ converges; certainly (x_n) converges to the same limit.

(b) This is true. Consider the contrapositive statement: if (x_n) converges, then all subsequences of (x_n) converge. This is implied by Theorem 2.5.2.

(c) This is true. Consider the sequences

$$a_n = \inf\{x_m : m \geq n\} \quad \text{and} \quad b_n = \sup\{x_m : m \geq n\}.$$

As shown in Exercise 2.4.7 these sequences both converge since (x_n) is bounded and their limits are denoted by

$$\liminf x_n = \lim a_n \quad \text{and} \quad \limsup x_n = \lim b_n.$$

We will construct a subsequence of (x_n) converging to $\limsup x_n$. Let $n_0 = 0$. Because b_1 is the supremum of the set $\{x_1, x_2, x_3, \ldots\}$, Lemma 1.3.8 implies that there exists an $n_1 \geq 1$ such that $b_1 - 1 < x_{n_1} \leq b_1$. Similarly, because b_{n_1+1} is the supremum of the set

$$\left\{x_{n_1+1}, x_{n_1+2}, x_{n_1+3}, \ldots\right\},$$

Lemma 1.3.8 gives us an $n_2 \geq n_1 + 1$ such that $b_{n_1+1} - \frac{1}{2} < x_{n_2} \leq b_{n_1+1}$. Continuing in this fashion, we obtain indices $n_1 < \cdots < n_k < \cdots$ such that

$$b_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq b_{n_{k-1}+1} \tag{$*$}$$

for each $k \in \mathbf{N}$. Notice that $(b_{n_{k-1}+1})_{k=1}^{\infty}$ is a subsequence of $(b_n)_{n=1}^{\infty}$, which converges to $\limsup x_n$; it follows from Theorem 2.5.2 that $(b_{n_{k-1}+1})_{k=1}^{\infty}$ also converges to $\limsup x_n$. The Squeeze Theorem (Exercise 2.3.3) and $(*)$ then imply that $\lim_{k\to\infty} x_{n_k} = \limsup x_n$. Similarly, we can find a subsequence of (x_n) converging to $\liminf x_n$. As we showed in Exercise 2.4.7, the fact that (x_n) diverges implies that $\liminf x_n < \limsup x_n$ and thus we have found two subsequences of (x_n) that converge to different limits.

(d) This is true. Suppose that (x_n) is decreasing; the case where (x_n) is increasing is handled similarly. By assumption there is a subsequence (x_{n_k}), which must also be decreasing, converging to some $x \in \mathbf{R}$. By the Monotone Convergence Theorem (Theorem 2.4.2) and the uniqueness of limits (Theorem 2.2.7), we have

$$\lim_{k\to\infty} x_{n_k} = x = \inf\left\{x_{n_k} : k \in \mathbf{N}\right\}.$$

Let $\varepsilon > 0$ be given. Since $x_{n_k} \to x$, there is a $K \in \mathbf{N}$ such that $\left|x_{n_K} - x\right| < \varepsilon$. Suppose that $n \in \mathbf{N}$ is such that $n \geq n_K$. Because (x_{n_k}) is a subsequence, there exists some $k \in \mathbf{N}$ such that $n_k \geq n$. Since (x_n) is decreasing, we then have

$$x \leq x_{n_k} \leq x_n \leq x_{n_K} < x + \varepsilon \quad \Rightarrow \quad |x_n - x| < \varepsilon.$$

Thus $\lim_{n\to\infty} x_n = x$.

Exercise 2.5.3.

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Solution.

(a) We have indices $n_1 < \cdots < n_k < \cdots$ and we want to show that $\sum_{k=1}^{\infty} b_k = L$, where $b_1 = a_1 + \cdots + a_{n_1} = s_{n_1}$ and

$$b_k = a_{n_{k-1}+1} + \cdots + a_{n_k} = s_{n_k} - s_{n_{k-1}}$$

for $k \geq 2$. Observe that for $m \geq 2$, the partial sums are

$$t_m = \sum_{k=1}^{m} b_k = s_{n_1} + \sum_{k=2}^{m} (s_{n_k} - s_{n_{k-1}})$$
$$= s_{n_1} + (s_{n_2} - s_{n_1}) + \cdots + (s_{n_m} - s_{n_{m-1}}) = s_{n_m}.$$

It follows from Theorem 2.5.2 that $\sum_{k=1}^{\infty} b_k = \lim_{m\to\infty} t_m = \lim_{m\to\infty} s_{n_m} = L$.

(b) Our proof does not apply to the series $\sum_{n=1}^{\infty} (-1)^n$ since this series does not converge: the sequence of partial sums is $(-1, 0, -1, 0, \ldots)$.

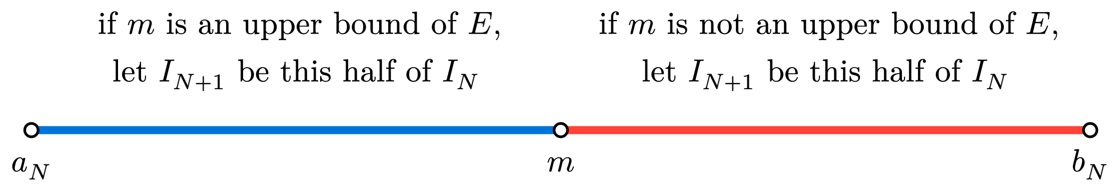
Exercise 2.5.4. The Bolzano-Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Solution. Let $E \subseteq \mathbf{R}$ be non-empty and bounded above by some $b_1 \in \mathbf{R}$. We will show that $\sup E$ exists. If E has a maximum x, then $\sup E = x$. Otherwise, we will inductively construct a sequence $(I_n)_{n=1}^{\infty}$ of nested intervals. E is non-empty, so pick some $a_1 \in E$; it must be the case that a_1 is not an upper bound of E since E has no maximum. Let $I_1 = [a_1, b_1]$.

Suppose that after N steps we have chosen intervals $I_n = [a_n, b_n]$ for $n \in \{1, \ldots, N\}$ such that

- $a_1 \leq \cdots \leq a_N$ are not upper bounds of E;
- $b_N \leq \cdots \leq b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for each $n \in \{1, \ldots, N\}$.

Let $m = \frac{a_N + b_N}{2}$ be the midpoint of the interval I_N. If m is an upper bound of E let $a_{N+1} = a_N$ and $b_{N+1} = m$, and if m is not an upper bound of E let $a_{N+1} = m$ and $b_{N+1} = b_N$; now let $I_{N+1} = [a_{N+1}, b_{N+1}]$.



In either case, we have chosen intervals $I_n = [a_n, b_n]$ for $n \in \{1, \ldots, N+1\}$ such that

- $a_1 \le \cdots \le a_{N+1}$ are not upper bounds of E;
- $b_{N+1} \le \cdots \le b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for each $n \in \{1, \ldots, N+1\}$.

This inductive process provides us with a sequence $(I_n)_{n=1}^{\infty}$ of intervals $I_n = [a_n, b_n]$ with the following properties:

- $(a_n)_{n=1}^{\infty}$ is an increasing sequence, the terms of which are not upper bounds of E;
- $(b_n)_{n=1}^{\infty}$ is a decreasing sequence, the terms of which are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for each $n \in \mathbf{N}$.

Because (a_n) is increasing and (b_n) is decreasing, the intervals (I_n) are nested. By assumption $\mathbf{R}$ has the Nested Interval Property (Theorem 1.4.1), so there exists an $x \in \mathbf{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$ we claim that $x = \sup E$. Let $y \in E$ and $\varepsilon > 0$ be given. Since $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for each $n \in \mathbf{N}$ and $(2^{-n}) \to 0$ (by assumption), there must exist an $N \in \mathbf{N}$ such that

$$|I_N| = b_N - a_N < \varepsilon \quad \Rightarrow \quad x + (b_N - a_N) < x + \varepsilon.$$

Because $x \in \bigcap_{n=1}^{\infty} I_n$ we then have

$$a_N \le x \quad \Rightarrow \quad b_N \le x + (b_N - a_N) \quad \Rightarrow \quad b_N < x + \varepsilon.$$

Since $y \in E$ and b_N is an upper bound of E, it follows that $y \le b_N < x + \varepsilon$. Thus $y < x + \varepsilon$ for every $\varepsilon > 0$; it follows from Exercise 1.2.10 (c) that $y \le x$. Because $y \in E$ was arbitrary, we see that x is an upper bound of E.

Now suppose that $t \in \mathbf{R}$ is such that $t < x$. Since $(|I_n|) \to 0$, there must be an $N \in \mathbf{N}$ such that

$$|I_N| = b_N - a_N < x - t \quad \Rightarrow \quad t < x - (b_N - a_N).$$

Because $x \in \bigcap_{n=1}^{\infty} I_n$ we then have

$$x \le b_N \quad \Rightarrow \quad x - (b_N - a_N) \le a_N \quad \Rightarrow \quad t < a_N.$$

It follows that t is not an upper bound of E since a_N is not an upper bound of E. We may conclude that x is the least upper bound of E, i.e. $x = \sup E$.

We had to assume that $(2^{-n}) \to 0$ since the usual proof of this would involve the Archimedean Property (Theorem 1.4.2), which we proved using the Axiom of Completeness.

Exercise 2.5.5. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Solution. Since (a_n) is bounded, $\liminf a_n$ and $\limsup a_n$ both exist. In the solution to Exercise 2.5.2 (c) we showed that there are subsequences (a_{n_k}) and (a_{n_ℓ}) such that

$$\lim_{k\to\infty} a_{n_k} = \liminf a_n \quad \text{and} \quad \lim_{\ell\to\infty} a_{n_\ell} \limsup a_n.$$

By assumption we have $\lim_{k\to\infty} a_{n_k} = \lim_{\ell\to\infty} a_{n_\ell} = a$ and so by the uniqueness of limits (Theorem 2.2.7) it follows that $\liminf a_n = \liminf a_n = a$. Exercise 2.4.7 then implies that $\lim a_n = a$.

Exercise 2.5.6. Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Solution. If $b = 0$ then $b^{1/n} = 0$ for all $n \in \mathbf{N}$ and thus $\lim b^{1/n} = 0$. Suppose that $b > 0$ and observe that

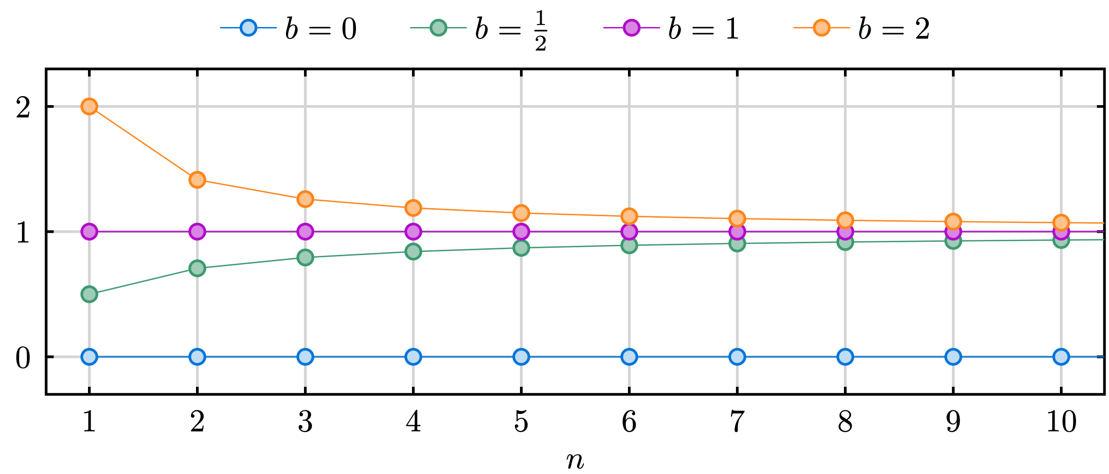
$$0 < b < 1 \quad \Rightarrow \quad b < b^{1/2} < b^{1/3} < \cdots < 1 \quad \text{and} \quad b \geq 1 \quad \Rightarrow \quad b \geq b^{1/2} \geq b^{1/3} \geq \cdots \geq 1.$$

In either case $\left(b^{1/n}\right)$ is bounded and monotone and hence convergent by the Monotone Convergence Theorem (Theorem 2.4.2), say $\lim b^{1/n} = L \in \mathbf{R}$. Note that, by Theorem 2.5.2, $\lim b^{1/2n} = L$ also. Note further that

$$\lim b^{1/2n} = \lim \sqrt{b^{1/n}} = \sqrt{\lim b^{1/n}} = \sqrt{L}$$

by Exercise 2.3.1. Since limits are unique (Theorem 2.2.7) we must have $L = \sqrt{L}$, which implies that $L = 0$ or $L = 1$. If $0 < b < 1$ then the Order Limit Theorem (Theorem 2.3.4) gives $0 < b < L \leq 1$, so that $L = 1$, and if $b \geq 1$ then the Order Limit Theorem gives $L \geq 1$ and thus $L = 1$.

We may conclude that $\lim b^{1/n} = 0$ if $b = 0$ and $\lim b^{1/n} = 1$ if $b > 0$.



Exercise 2.5.7. Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Solution. Consider the following cases.

Case 1. $b > 1$. In this case (b^n) is unbounded and hence divergent.

Case 2. $b = 1$. In this case $(b^n) = (1, 1, 1, ...)$ and thus $\lim b^n = 1$.

Case 3. $0 < b < 1$. Example 2.5.3 shows that in this case we have $\lim b^n = 0$.

Case 4. $b = 0$. In this case $(b^n) = (0, 0, 0, ...)$ and thus $\lim b^n = 0$.

Case 5. $-1 < b < 0$. Observe that $b = (-1)|b|$, so that $b^n = (-1)^n |b|^n$. Since $0 < |b| < 1$, we have $\lim |b|^n = 0$ by the $0 < b < 1$ case. Given this, and the boundedness of $(-1)^n$, it follows from Exercise 2.3.9 (a) that

$$\lim b^n = \lim\left[(-1)^n |b|^n\right] = 0.$$

Case 6. $b = -1$. In this case $b^n = (-1)^n$, which is divergent since it has two convergent subsequences with different limits:

$$\lim\left[(-1)^{2n}\right] = 1 \neq -1 = \lim\left[(-1)^{2n+1}\right].$$

Case 7. $b < -1$. We have $b^n = (-1)^n |b|^n$ with $|b| > 1$. Observe that the subsequence $(b^{2n}) = \left(|b|^{2n}\right)$ is divergent by the $b > 1$ case. It then follows from Exercise 2.5.2 (b) that the sequence (b^n) is divergent.

We may conclude that $\lim b^n = 0$ if and only if $-1 < b < 1$.

Exercise 2.5.8. Another way to prove the Bolzano-Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term*. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

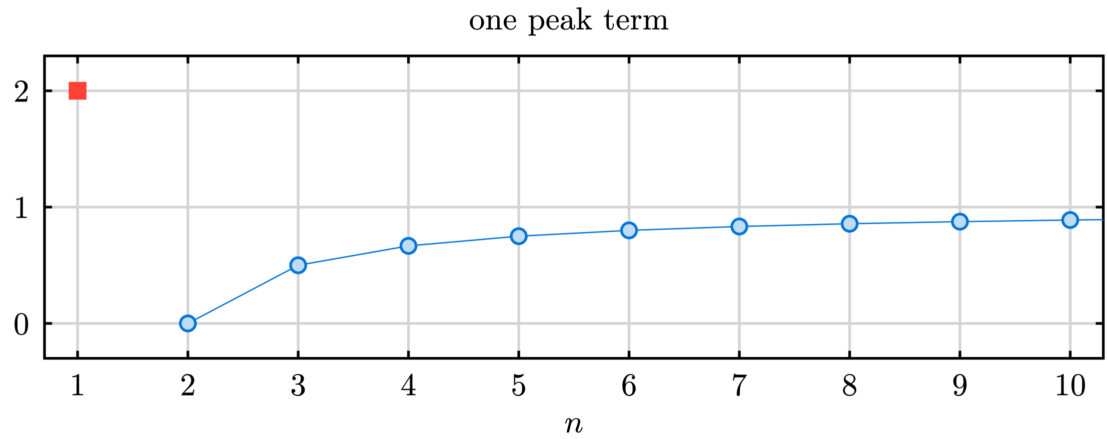
(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.

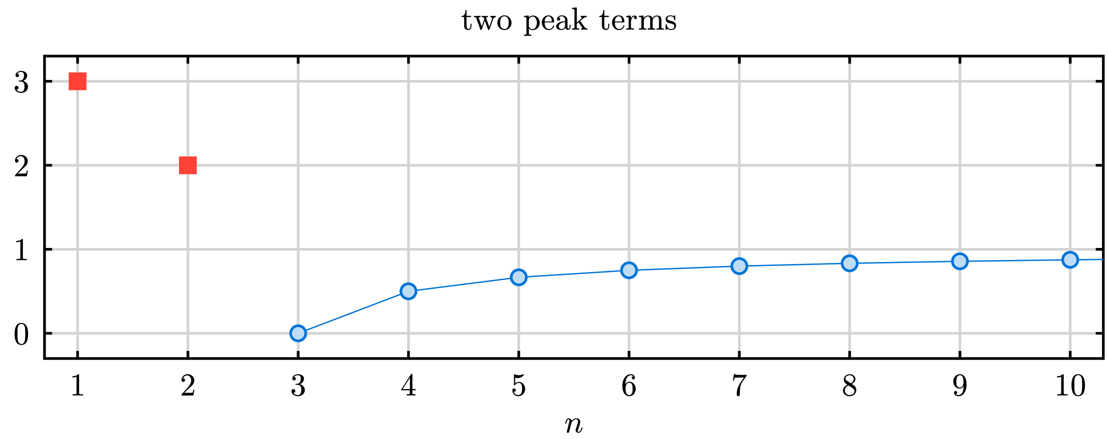
(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.

Solution.

(a) Any strictly increasing sequence will have zero peak terms; the sequence $(1, 2, 3, ...)$ for example. For sequences with one and two peak terms, consider

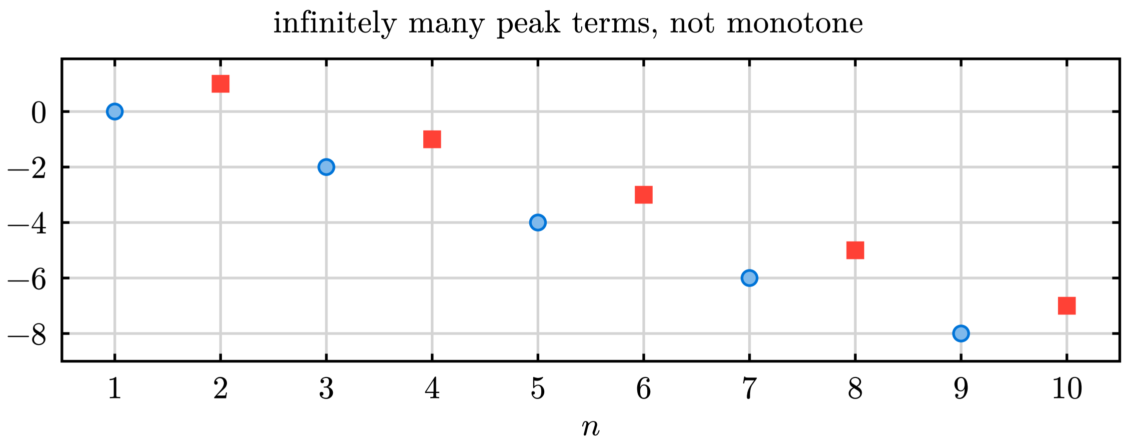
$$\left(2, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...\right) \quad \text{and} \quad \left(3, 2, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...\right).$$





For a sequence with infinitely many peak terms but which is not monotone, consider

$$(0, 1, -2, -1, -4, -3, -6, -5, \ldots).$$



(b) Let (x_n) be a sequence. If (x_n) contains infinitely many peak terms $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$, where we may assume that $n_1 < n_2 < \cdots < n_k < \cdots$, then the subsequence (x_{n_k}) is a decreasing subsequence of (x_n). If (x_n) contains only finitely many peak terms, then we are guaranteed the existence of a term x_{n_1} which is not a peak term and after which there are no peak terms. Since x_{n_1} is not a peak term, there exists an $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$ and x_{n_2} is not a peak term. Since x_{n_2} is not a peak term, there exists an $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$ and x_{n_3} is not a peak term. Continuing in this way, we obtain an increasing subsequence (x_{n_k}).

Now suppose that (x_n) is a bounded sequence. By the previous paragraph there exists a monotone subsequence (x_{n_k}), which must also be bounded. The Monotone Convergence Theorem (Theorem 2.4.2) then implies that (x_{n_k}) is convergent. This provides another proof of the Bolzano-Weierstrass Theorem.

Exercise 2.5.9. Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Solution. Since (a_n) is bounded, there is an $M > 0$ such that $-M \leq a_n \leq M$ for all $n \in \mathbf{N}$. It follows that $(-\infty, -M) \subseteq S$, so that S is non-empty, and for any $x \in S$ we have $x < a_n \leq M$ for some $n \in \mathbf{N}$, so that S is bounded above by M. Thus, by the Axiom of Completeness, $s = \sup S$ exists in $\mathbf{R}$.

Let k be a positive integer. We claim that the set

$$C_k = \left\{n \in \mathbf{N} : s - \frac{1}{k} < a_n \leq s + \frac{1}{k}\right\}$$

is infinite. By Lemma 1.3.8 there exists an $x \in S$ such that $s - \frac{1}{k} < x \leq s$. Define the sets

$$E = \{n \in \mathbf{N} : x < a_n\}, \quad A_k = \left\{n \in \mathbf{N} : s + \frac{1}{k} < a_n\right\}, \quad B_k = \left\{n \in \mathbf{N} : x < a_n \leq s + \frac{1}{k}\right\}.$$

Observe that E is the disjoint union of A_k and B_k and that E is infinite since $x \in S$. Furthermore, A_k must be finite, otherwise we would have $s + \frac{1}{k} \in S$. It follows that B_k is infinite and hence that C_k is infinite, since $B_k \subseteq C_k$.

Since C_1 is infinite, there exists some $n_1 \in \mathbf{N}$ such that $s - 1 < a_{n_1} \leq s + 1$. Since C_2 is infinite, there exists some $n_2 > n_1$ such that $s - \frac{1}{2} < a_{n_2} \leq s + \frac{1}{2}$. Continuing this process, we obtain a subsequence (a_{n_k}) satisfying $s - \frac{1}{k} < a_{n_k} \leq s + \frac{1}{k}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim_{k\to\infty} a_{n_k} = s$.

2.6. The Cauchy Criterion

Exercise 2.6.1. Supply a proof for Theorem 2.6.2.

Solution. Suppose $x_n \to x$ for some $x \in \mathbf{R}$ and let $\varepsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ whenever $n \geq N$. For $m, n \geq N$ we then have

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (x_n) is Cauchy.

Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

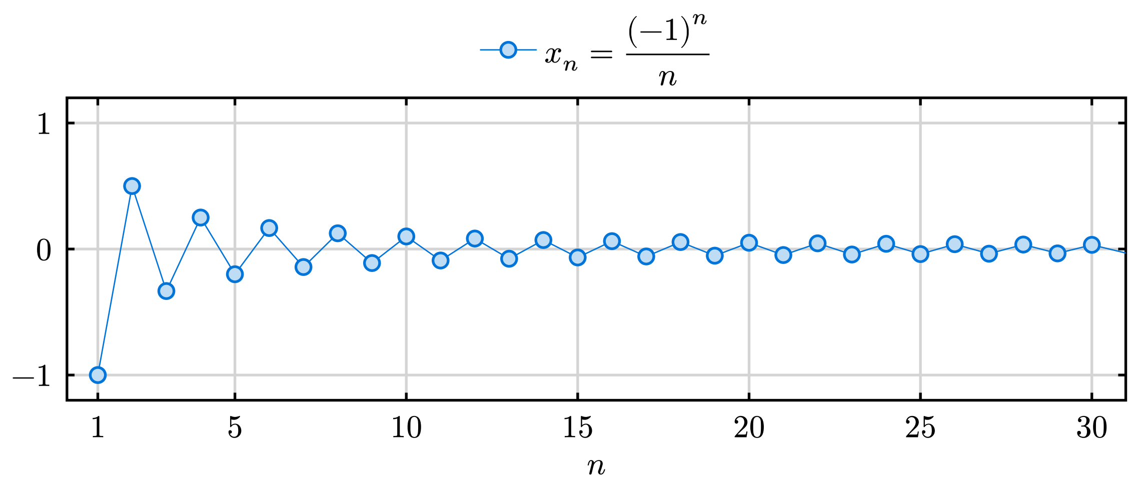
(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution.

(a) Consider the sequence (x_n) given by $x_n = \frac{(-1)^n}{n}$. The sequence is convergent ($\lim x_n = 0$) and hence Cauchy (Theorem 2.6.4), but is certainly not monotone.



(b) This is impossible. A Cauchy sequence (x_n) is necessarily convergent (Theorem 2.6.4) and hence all subsequences of (x_n) must be convergent (Theorem 2.5.2); each subsequence must then be bounded (Theorem 2.3.2).

(c) First, let us prove the following result.

Lemma L.7. If (x_n) is an unbounded monotone sequence then all subsequences of (x_n) are also unbounded and monotone.

Proof. Suppose (x_n) is increasing (the case where (x_n) is decreasing is handled similarly) and let (x_{n_k}) be a subsequence of (x_n). If $k > \ell$ then $n_k > n_\ell$ and thus $x_{n_k} \geq x_{n_\ell}$ since (x_n) is increasing; it follows that (x_{n_k}) is an increasing sequence. Now let $M > 0$ be given. Since (x_n) is unbounded, there is an $N \in \mathbf{N}$ such that $x_N > M$, and since (x_{n_k}) is a subsequence of (x_n) we are guaranteed the existence of a $K \in \mathbf{N}$ such that $n_K > N$; it follows that $x_{n_K} \geq x_N > M$ since (x_n) is increasing. We may conclude that (x_{n_k}) is unbounded. □

We can now show that the given request is impossible. If (x_n) is a divergent monotone sequence then by the Monotone Convergence Theorem (Theorem 2.4.2) the sequence (x_n) must be unbounded. It follows from Lemma L.7 that all subsequences of (x_n) are unbounded, hence divergent (Theorem 2.3.2), and hence not Cauchy (Theorem 2.6.4).

(d) Consider the unbounded sequence $(0, 1, 0, 2, 0, 3, \ldots)$. The subsequence $(0, 0, 0, \ldots)$ is convergent and hence Cauchy (Theorem 2.6.4).

Exercise 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Solution.

(a) Let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$m, n \geq N_1 \quad \Rightarrow \quad |x_n - x_m| < \frac{\varepsilon}{2} \quad \text{and} \quad m, n \geq N_2 \quad \Rightarrow \quad |y_n - y_m| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $m, n \geq N$ we have

$$|x_n + y_n - x_m - y_m| \leq |x_n - x_m| + |y_n - y_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that $(x_n + y_n)$ is a Cauchy sequence.

(b) Because Cauchy sequences are bounded (Lemma 2.6.3), there are positive real numbers M_1 and M_2 such that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all $n \in \mathbf{N}$. Let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$m, n \geq N_1 \quad \Rightarrow \quad |x_n - x_m| < \frac{\varepsilon}{2M_2} \quad \text{and} \quad m, n \geq N_2 \quad \Rightarrow \quad |y_n - y_m| < \frac{\varepsilon}{2M_1}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $m, n \geq N$ we have

$$|x_n y_n - x_m y_m| = |x_n y_n - x_m y_n + x_m y_n - x_m y_m|$$
$$\leq |y_n||x_n - x_m| + |x_m||y_n - y_m| < M_2 \frac{\varepsilon}{2M_2} + M_1 \frac{\varepsilon}{2M_1} = \varepsilon.$$

It follows that $(x_n y_n)$ is a Cauchy sequence.

Exercise 2.6.4. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution. By the Cauchy Criterion (Theorem 2.6.4), we have $\lim a_n = a$ and $\lim b_n = b$ for some real numbers a and b. Again by the Cauchy Criterion, it will suffice to consider convergence of the given sequences (c_n).

(a) By Exercise 2.3.10 (b) and the Algebraic Limit Theorem (Theorem 2.3.3), we have

$$\lim c_n = \lim|a_n - b_n| = |\lim a_n - \lim b_n| = |a - b|.$$

Thus (c_n) is convergent and hence Cauchy.

(b) Suppose that $a = 0$. By Exercise 2.3.9 (a) we then have $\lim c_n = 0$ and it follows that (c_n) is Cauchy. If $a \neq 0$ then observe that

$$\lim c_{2n} = \lim a_{2n} = a \neq -a = \lim(-a_{2n-1}) = \lim(c_{2n-1}).$$

Thus (c_n) has two subsequences which converge to different limits. It follows that (c_n) is not convergent (Theorem 2.5.2) and hence not Cauchy.

(c) Suppose that a is not an integer, so that $[[a]] < a < [[a]] + 1$. Let

$$\delta = \min\{a - [[a]], [[a]] + 1 - a\}.$$

Since $\lim a_n = a$, there is a positive integer N such that $a_n \in (a - \delta, a + \delta)$ whenever $n \geq n$. Observe that $[[a]] \leq a - \delta$ and $a + \delta \leq [[a]] + 1$. For $n \geq N$ we then have $[[a]] < a_n < [[a]] + 1$, which gives us $[[a_n]] = [[a]]$. Thus the sequence $[[a_n]]$ is eventually constant with value $[[a]]$; it follows that $[[a_n]]$ is convergent with limit $[[a]]$ and hence Cauchy.

If a is an integer then the sequence $([[a_n]])$ may or may not be convergent (and so may or may not be Cauchy). For example, if (a_n) is the sequence $(0, 0, 0, \ldots)$ then $\lim[[a_n]] = 0$. However, consider the sequence $a_n = \frac{(-1)^n}{n}$, which also satisfies $\lim a_n = 0$. This gives

$$([[a_n]]) = (-1, 0, -1, 0, -1, 0, \ldots),$$

which is divergent.

Exercise 2.6.5. Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \varepsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution.

(i) This statement is false: consider the sequence (s_n) given by $s_n = \sum_{m=1}^{n} \frac{1}{m}$. This sequence satisfies $s_{n+1} - s_n = \frac{1}{n+1} \to 0$, so that (s_n) is pseudo-Cauchy. However, as shown in Example 2.4.5, (s_n) is unbounded.

(ii) This statement is true. Let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \quad \Rightarrow \quad |x_{n+1} - x_n| < \frac{\varepsilon}{2} \quad \text{and} \quad n \geq N_2 \quad \Rightarrow \quad |y_{n+1} - y_n| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$|x_{n+1} + y_{n+1} - x_n - y_n| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(x_n + y_n)$ is pseudo-Cauchy.

Exercise 2.6.6. Let's call a sequence (a_n) *quasi-increasing* if for all $\varepsilon > 0$ there exists an N such that whenever $n > m \geq n$ it follows that $a_n > a_m - \varepsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

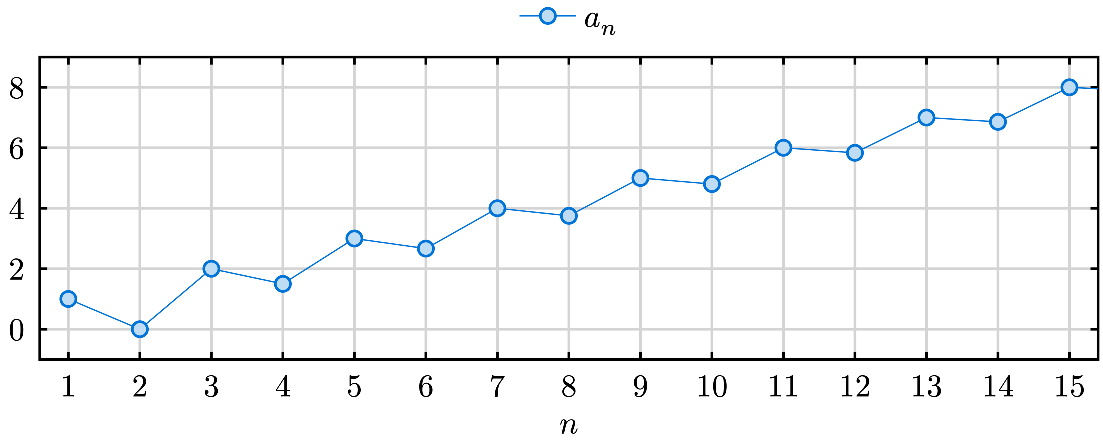
(b) Give an example of a quasi-increasing sequence that is divergent but not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? Give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

Solution.

(a) Consider the sequence (a_n) given by

$$a_n = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \frac{2}{n} & \text{if } n \text{ is even.} \end{cases}$$



Some calculations reveal that this sequence has the following properties.

(i) If $m \in \mathbf{N}$ is even then $a_n > a_m$ for all $n > m$.

(ii) If $m \in \mathbf{N}$ is odd then $a_n > a_m$ for all $n > m+1$ and $a_m - a_{m+1} = \frac{2}{m+1} > 0$.

It follows that (a_n) is not eventually monotone, for if N is a positive integer, choose an odd integer m such that $m > N$; by property (ii) we then have $a_m > a_{m+1}$ and $a_m < a_{m+2}$. Furthermore, (a_n) is quasi-increasing. Indeed, let $\varepsilon > 0$ be given, choose a positive integer N such that $\frac{2}{N+1} < \varepsilon$, and suppose that $n > m \geq N$. By properties (i) and (ii) we have

$$a_m - a_n < 0 < \varepsilon \quad \Rightarrow \quad a_n > a_m - \varepsilon,$$

unless m is odd and $n = m+1$. In that case we have

$$a_m - a_{m+1} = \frac{2}{m+1} \leq \frac{2}{N+1} < \varepsilon \quad \Rightarrow \quad a_n > a_m - \varepsilon.$$

(b) The sequence (a_n) given in part (a) is unbounded and hence divergent.

(c) There is an analogue of the Monotone Convergence Theorem for bounded quasi-increasing sequences. Let (a_n) be such a sequence. We will show that (a_n) converges to $\limsup a_n$.

Let $s = \limsup a_n$ and $y_n = \sup\{a_\ell : \ell \geq n\}$, so that $\lim y_n = s$. By Exercise 2.5.2 (c) there is a subsequence (a_{n_k}) converging to s. Let $\varepsilon > 0$ be given. There is an $N_1 \in \mathbf{N}$ such that $|y_n - s| < \varepsilon$ whenever $n \geq N_1$. Since $a_n \leq y_n$ for all $n \in \mathbf{N}$, we have

$$n \geq N_1 \quad \Rightarrow \quad a_n < s + \varepsilon. \tag{1}$$

Because (a_n) is quasi-increasing, there is an $N_2 \in \mathbf{N}$ such that

$$n > m \geq N_2 \quad \Rightarrow \quad a_m - \frac{\varepsilon}{2} < a_n, \tag{2}$$

and since $(a_{n_k}) \to s$, there is a $M \in \mathbf{N}$ such that

$$k \geq M \quad \Rightarrow \quad \left|a_{n_k} - s\right| < \frac{\varepsilon}{2}. \tag{3}$$

Because (a_{n_k}) is a subsequence, there must be some $K \in \mathbf{N}$ such that $K \geq M$ and $n_K \geq N_2$. It follows from (2) that

$$n > n_K \quad \Rightarrow \quad a_{n_K} - \frac{\varepsilon}{2} < a_n,$$

and it follows from (3) that $s - \varepsilon < a_{n_K} - \frac{\varepsilon}{2}$. Combining these gives

$$n > n_K \quad \Rightarrow \quad s - \varepsilon < a_n. \tag{4}$$

Let $N = \max\{N_1, n_K\}$. By (1) and (4) we then have

$$n > N \quad \Rightarrow \quad s - \varepsilon < a_n < s + \varepsilon.$$

Thus $\lim a_n = s$.

Exercise 2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show that the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Solution.

(a) Suppose (x_n) is bounded and increasing (the case where (x_n) is decreasing is handled similarly). By assumption there is a convergent subsequence (x_{n_k}), say $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in \mathbf{R}$. Let $\varepsilon > 0$ be given. There is a $K \in \mathbf{N}$ such that

$$k \geq K \quad \Rightarrow \quad \left|x_{n_k} - x\right| < \varepsilon. \tag{1}$$

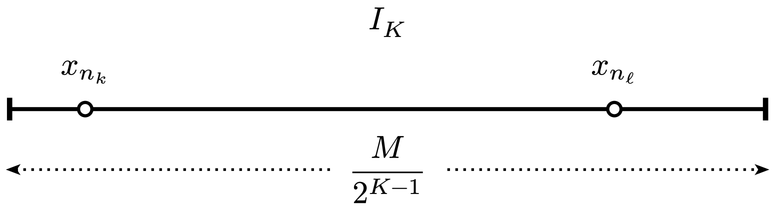
Suppose $n \in \mathbf{N}$ is such that $n \geq n_K$. Because (x_n) is increasing we then have $x - \varepsilon < x_{n_K} \leq x_n$. Furthermore, it must be the case that $x_n < x + \varepsilon$. Indeed, if $x_n \geq x + \varepsilon$ then since (x_{n_k}) is a subsequence there must be some $k \in \mathbf{N}$ such that $n_k \geq n \geq n_K$. This implies that $k \geq K$ and, since (x_n) is increasing, that $x_{n_k} \geq x_n \geq x + \varepsilon$ this contradicts (1). Thus we have

$$n \geq n_K \quad \Rightarrow \quad x - \varepsilon < x_n < x + \varepsilon.$$

It follows that $\lim x_n = x$.

(b) Let (x_n) be a sequence bounded by some $M > 0$. As in the proof of the Bolzano-Weierstrass Theorem (Theorem 2.5.5) given in the textbook, construct a sequence of nested intervals (I_k) with length $M \cdot 2^{-k+1}$ and a subsequence (x_{n_k}) such that $x_{n_k} \in I_k$. Let

$\varepsilon > 0$ be given. Assuming that $2^{-k} \to 0$ (this is equivalent to assuming the Archimedean Property), there is a $K \in \mathbf{N}$ such that $M \cdot 2^{-K+1} < \varepsilon$. Suppose that $k > \ell \geq K$. Since the intervals are nested, both x_{n_k} and x_{n_ℓ} belong to I_K.



It follows that x_{n_k} and x_{n_ℓ} are no further apart than the length of I_K, i.e.

$$\left|x_{n_k} - x_{n_\ell}\right| \leq \frac{M}{2^{K-1}} < \varepsilon.$$

This demonstrates that (x_{n_k}) is a Cauchy sequence. By assumption this is equivalent to (x_{n_k}) being convergent.

(c) The ordered field $\mathbf{Q}$ has the Archimedean Property but does not satisfy the Axiom of Completeness (see Lemma L.4; the subset $A \subseteq \mathbf{Q}$ given there is non-empty and bounded above but has no supremum in $\mathbf{Q}$).

2.7. Properties of Infinite Series

Exercise 2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Solution. First note that since (a_n) is decreasing and converges to zero, $a_n \geq 0$ and $a_n - a_{n+1} \geq 0$ for all $n \in \mathbf{N}$.

(a) Suppose $n > m$ are positive integers. If $n - m$ is even then

$$\underbrace{a_{m+1} - a_{m+2}}_{\geq 0} + \underbrace{a_{m+3} - a_{m+4}}_{\geq 0} + \cdots + \underbrace{a_{n-1} - a_n}_{\geq 0} \geq 0,$$

$$\text{and} \quad a_{m+1} + \underbrace{(-a_{m+2} + a_{m+3})}_{\leq 0} + \cdots + \underbrace{(-a_{n-2} + a_{n-1})}_{\leq 0} + \underbrace{(-a_n)}_{\leq 0} \leq a_{m+1}.$$

If $n - m$ is odd then

$$\underbrace{a_{m+1} - a_{m+2}}_{\geq 0} + \underbrace{a_{m+3} - a_{m+4}}_{\geq 0} + \cdots + \underbrace{a_{n-2} - a_{n-1}}_{\geq 0} + \underbrace{a_n}_{\geq 0} \geq 0,$$

$$\text{and} \quad a_{m+1} + \underbrace{(-a_{m+2} + a_{m+3})}_{\leq 0} + \cdots + \underbrace{(-a_{n-1} + a_n)}_{\leq 0} \leq a_{m+1}.$$

It follows that

$$|s_n - s_m| = a_{m+1} - a_{m+2} + \cdots \pm a_n \leq a_{m+1}.$$

Let $\varepsilon > 0$ be given. Because $a_n \to 0$, there is an $N \in \mathbf{N}$ such that $|a_n| = a_n < \varepsilon$ for all $n \geq N$. For $n > m \geq N$ we then have

$$|s_n - s_m| \leq a_{m+1} < \varepsilon.$$

Thus (s_n) is a Cauchy sequence.

(b) Let n be a positive integer and observe that

$$s_{2n-1} - s_{2n} = a_{2n} \geq 0 \quad \Rightarrow \quad s_{2n} \leq s_{2n-1},$$
$$s_{2n-1} - s_{2n-3} = a_{2n-1} - a_{2n-2} \leq 0 \quad \Rightarrow \quad s_{2n-1} \leq s_{2n-3},$$
$$s_{2n} - s_{2n-2} = a_{2n-1} - a_{2n} \geq 0 \quad \Rightarrow \quad s_{2n-2} \leq s_{2n}.$$

Thus $(I_n = [s_{2n}, s_{2n-1}])_{n=1}^{\infty}$ is a sequence of nested intervals. It follows from the Nested Interval Property (Theorem 1.4.1) that there exists some $x \in \bigcap_{n=1}^{\infty} I_n$; we claim that $\lim s_n = x$. Suppose that $n \in \mathbf{N}$. If n is even then $s_n \in I_{n/2} = [s_n, s_{n-1}]$ and thus

$$|s_n - x| \leq \left|I_{n/2}\right| = s_{n-1} - s_n = a_n.$$

If n is odd then $s_n \in I_{(n+1)/2} = [s_{n+1}, s_n]$ and thus

$$|s_n - x| \leq \left|I_{(n+1)/2}\right| = s_n - s_{n+1} = a_{n+1} \leq a_n.$$

It follows that $|s_n - x| \leq a_n$ for all $n \in \mathbf{N}$. Since $a_n \to 0$, an application of the Squeeze Theorem (Exercise 2.3.3) then yields $\lim s_n = x$.

(c) As shown in part (b), the sequence (s_{2n}) is increasing and bounded above by s_1, and the sequence (s_{2n+1}) is decreasing and bounded below by s_2. The Monotone Convergence Theorem (Theorem 2.4.2) then implies that $\lim s_{2n}$ and $\lim s_{2n+1}$ both exist. Observe that

$$\lim(s_{2n+1} - s_{2n}) = \lim a_{2n+1} = 0,$$

so that s_{2n} and s_{2n+1} both converge to the same limit $x \in \mathbf{R}$ (Exercise 2.3.10 (c)). It follows that $\lim s_n = x$, as the next lemma shows.

Lemma L.8. If (x_n) is a sequence such that

$$\lim x_{2n} = \lim x_{2n+1} = x$$

for some $x \in \mathbf{R}$, then $\lim x_n = x$.

Proof. Let $\varepsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \quad \Rightarrow \quad |x_{2n} - x| < \varepsilon, \tag{1}$$
$$n \geq N_2 \quad \Rightarrow \quad \left|x_{2n+1} - x\right| < \varepsilon. \tag{2}$$

Let $N = \max\{N_1, N_2\}$ and suppose that $n \in \mathbf{N}$ is such that $n \geq 2N + 1$. If n is even then $\frac{n}{2} > N \geq N_1$ and so $|x_n - x| < \varepsilon$ by (1). If n is odd then $\frac{n-1}{2} \geq N \geq N_2$ and so $|x_n - x| < \varepsilon$ by (2). Thus

$$n \geq 2N + 1 \quad \Rightarrow \quad |x_n - x| < \varepsilon.$$

Thus $\lim x_n = x$. □

Exercise 2.7.2. Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ (b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

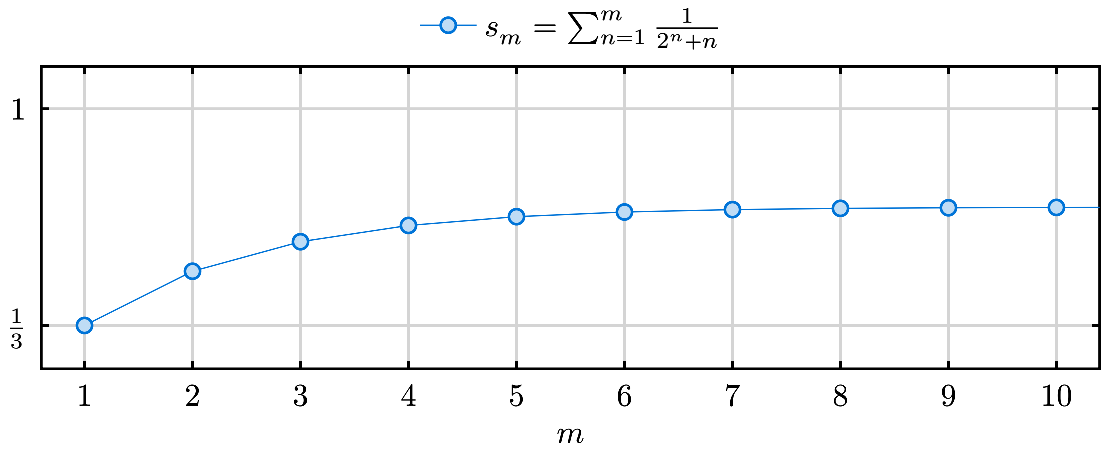
(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Solution.

(a) Observe that for each $n \in \mathbf{N}$ we have

$$0 < \frac{1}{2^n + n} < \frac{1}{2^n}.$$

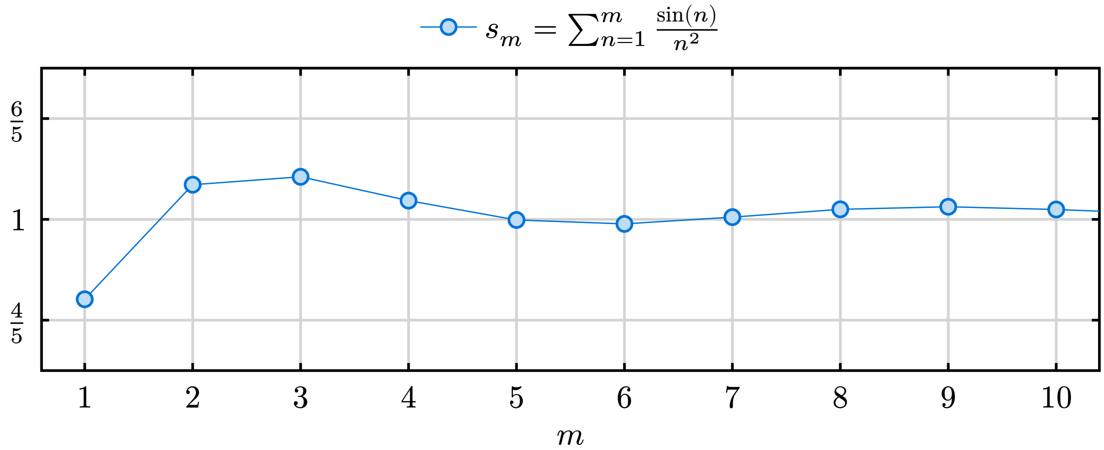
Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ (Example 2.7.5), the Comparison Test (Theorem 2.7.4) implies that $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ is convergent.



(b) Observe that for each $n \in \mathbf{N}$ we have

$$0 < \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Example 2.4.4), the Comparison Test (Theorem 2.7.4) implies that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent and hence convergent (Theorem 2.7.6).



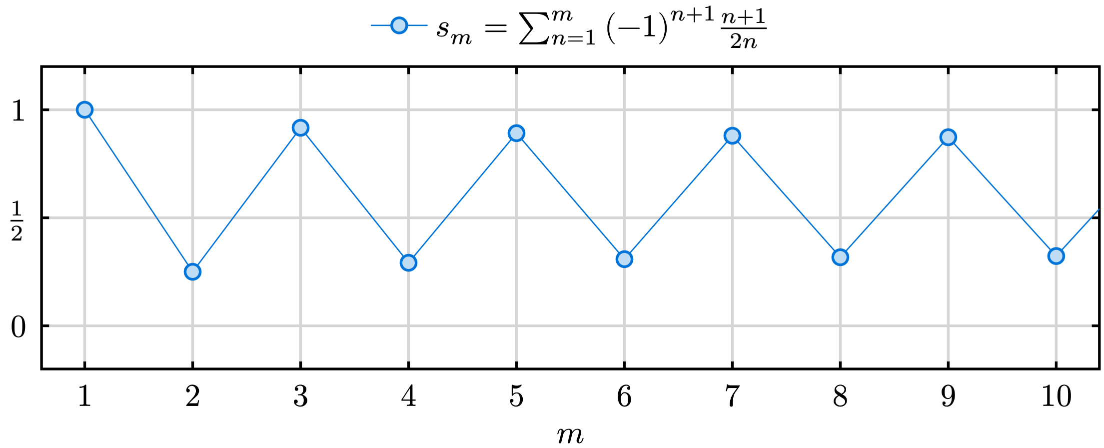
(c) This is the series $\sum_{n=1}^{\infty} a_n$, where

$$a_n = (-1)^{n+1}\frac{n+1}{2n} = (-1)^{n+1}\left(\frac{1}{2} + \frac{1}{2n}\right).$$

This sequence is divergent by Theorem 2.5.2:

$$\lim a_{2n} = -\frac{1}{2} \neq \frac{1}{2} = \lim a_{2n+1}.$$

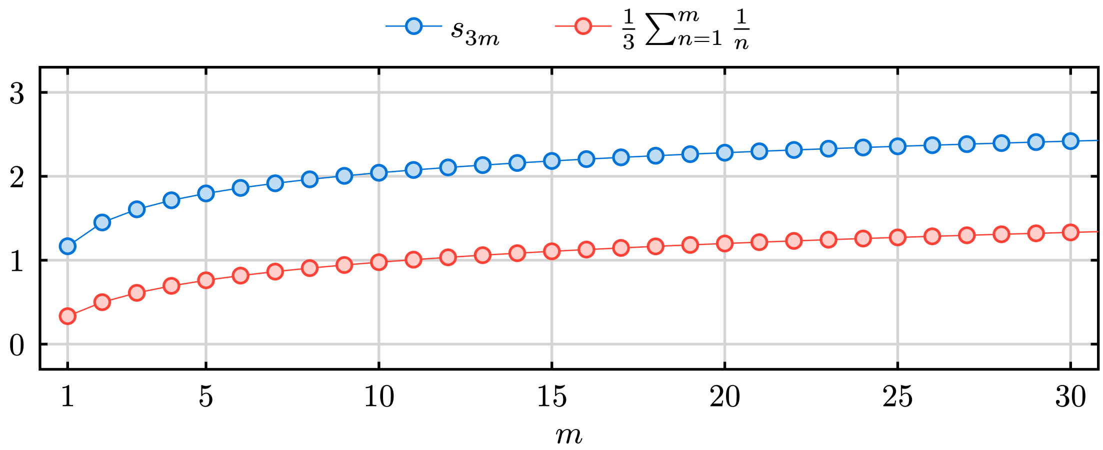
It follows from Theorem 2.7.3 that $\sum_{n=1}^{\infty} a_n$ is divergent.



(d) For the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$, let (s_m) be the sequence of partial sums and consider the subsequence (s_{3m}). Observe that

$$\begin{aligned}
s_{3m} &= \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{3m-2} + \frac{1}{3m-1} - \frac{1}{3m}\right) \\
&\geq \left(1 + \frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{3m-2} + \frac{1}{3m-1} - \frac{1}{3m-1}\right) \\
&= 1 + \frac{1}{4} + \cdots + \frac{1}{3m-2} \\
&= \frac{1}{3}\sum_{n=1}^{m} \frac{1}{n - \frac{2}{3}} \\
&\geq \frac{1}{3}\sum_{n=1}^{m} \frac{1}{n}.
\end{aligned}$$

So we have shown that $s_{3m} \geq \frac{1}{3}\sum_{n=1}^{m} \frac{1}{n}$ for all $m \in \mathbf{N}$. Since $\sum_{n=1}^{m} \frac{1}{n}$ is unbounded in m (Example 2.4.5), it follows that (s_{3m}) is unbounded. This implies that (s_m) is unbounded and hence divergent (Theorem 2.3.2).



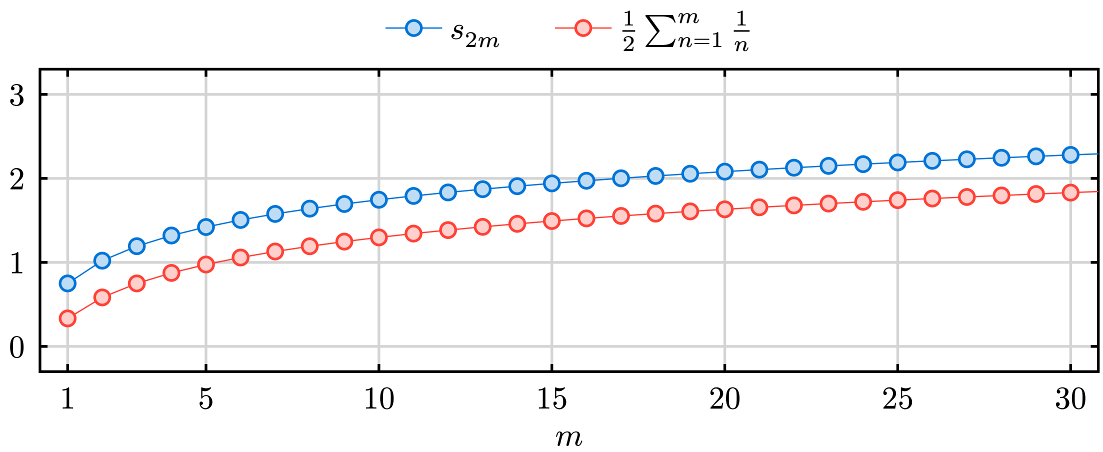
(e) For the series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$, let (s_m) be the sequence of partial sums and consider the subsequence (s_{2m}). For any $n \geq 2$ we have

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \quad \Rightarrow \quad -\frac{1}{n^2} \geq -\frac{1}{n-1} + \frac{1}{n}.$$

It follows that

$$\begin{aligned} s_{2m} &= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{3} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{2m-1} - \frac{1}{(2m)^2}\right) \\ &\geq \left(1 - 1 + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2m-1} - \frac{1}{2m-1} + \frac{1}{2m}\right) \\ &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m} \\ &= \frac{1}{2}\sum_{n=1}^{m}\frac{1}{n}. \end{aligned}$$

So we have shown that $s_{2m} \geq \frac{1}{2}\sum_{n=1}^{m}\frac{1}{n}$ for all $m \in \mathbf{N}$. Since $\sum_{n=1}^{m}\frac{1}{n}$ is unbounded in m (Example 2.4.5), it follows that (s_{2m}) is unbounded. This implies that (s_m) is unbounded and hence divergent (Theorem 2.3.2).



Exercise 2.7.3.

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Solution.

(a) Since $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$, for any $n > m$ we have

$$\left|a_{m+1} + \cdots + a_n\right| = a_{m+1} + \cdots + a_n \leq b_{m+1} + \cdots + b_n = \left|b_{m+1} + \cdots + b_n\right|. \tag{1}$$

Suppose that $\sum_{k=1}^{\infty} b_k$ is convergent and let $\varepsilon > 0$ be given. By the Cauchy Criterion for Series (Theorem 2.7.2), there exists an $N \in \mathbf{N}$ such that

$$n > m \geq N \quad \Rightarrow \quad \left|b_{m+1} + \cdots + b_n\right| < \varepsilon.$$

It then follows from inequality (1) that $\left|a_{m+1} + \cdots + a_n\right| < \varepsilon$ for all $n > m \geq N$. The Cauchy Criterion for Series allows us to conclude that $\sum_{k=1}^{\infty} a_k$ is convergent.

Now suppose that $\sum_{k=1}^{\infty} a_k$ is divergent. By the Cauchy Criterion for Series, there must exist an $\varepsilon > 0$ such that for all $N \in \mathbf{N}$ there are positive integers n and m such that

$$n > m \geq N \quad \text{and} \quad \left|a_{m+1} + \cdots + a_n\right| \geq \varepsilon.$$

Let $N \in \mathbf{N}$ be given and let n and m be the positive integers obtained above. Inequality (1) then gives us $\left|b_{m+1} + \cdots + b_n\right| \geq \varepsilon$; it follows from the Cauchy Criterion for Series that $\sum_{k=1}^{\infty} b_k$ is divergent.

(b) Define the sequences of partial sums

$$s_n = a_1 + \cdots + a_n \quad \text{and} \quad t_n = b_1 + \cdots + b_n.$$

Since $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$, both sequences of partial sums are increasing and satisfy $0 \leq s_n \leq t_n$ for all $n \in \mathbf{N}$. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that the convergence of each sequence is equivalent to the boundedness of that sequence. From the inequality $0 \leq s_n \leq t_n$, it is clear that (s_n) is bounded if (t_n) is bounded and that (t_n) is unbounded if (s_n) is unbounded.

Exercise 2.7.4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \le x_n \le 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution.

(a) If we let (x_n) and (y_n) be the sequences given by $x_n = y_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are both the divergent harmonic series (Example 2.4.5), but $\sum_{n=1}^{\infty} x_n y_n$ is the convergent (by Example 2.4.4) series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(b) Let (x_n) be the sequence given by $x_n = \frac{(-1)^{n+1}}{n}$ and (y_n) be the bounded sequence given by $y_n = (-1)^{n+1}$. It then follows from the Alternating Series Test (Theorem 2.7.7) that $\sum_{n=1}^{\infty} x_n$ is convergent, but $\sum_{n=1}^{\infty} x_n y_n$ is the divergent harmonic series.

(c) This is impossible. By Theorem 2.7.1 we must have

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} (x_n + y_n) - \sum_{n=1}^{\infty} x_n.$$

(d) Let (x_n) be the sequence given by

$$x_n = \begin{cases} \frac{1}{2(n+1)} & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even,} \end{cases} \qquad \text{i.e. } (x_n) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{12}, \frac{1}{6}, \dots\right),$$

and let (s_n) be the sequence of partial sums for the series $\sum_{n=1}^{\infty} (-1)^n x_n$. Note that $0 \le x_n \le \frac{1}{n}$ for all $n \in \mathbf{N}$. Note further that

$$\begin{aligned} s_{2m} &= \left(-\frac{1}{4} + \frac{1}{2}\right) + \left(-\frac{1}{8} + \frac{1}{4}\right) + \dots + \left(-\frac{1}{4m} + \frac{1}{2m}\right) \\ &= \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{4m} \\ &= \frac{1}{4} \sum_{n=1}^{m} \frac{1}{n}. \end{aligned}$$

It follows that (s_{2m}) is unbounded (Example 2.4.5) and hence that $\sum_{n=1}^{\infty} (-1)^n x_n$ is divergent.

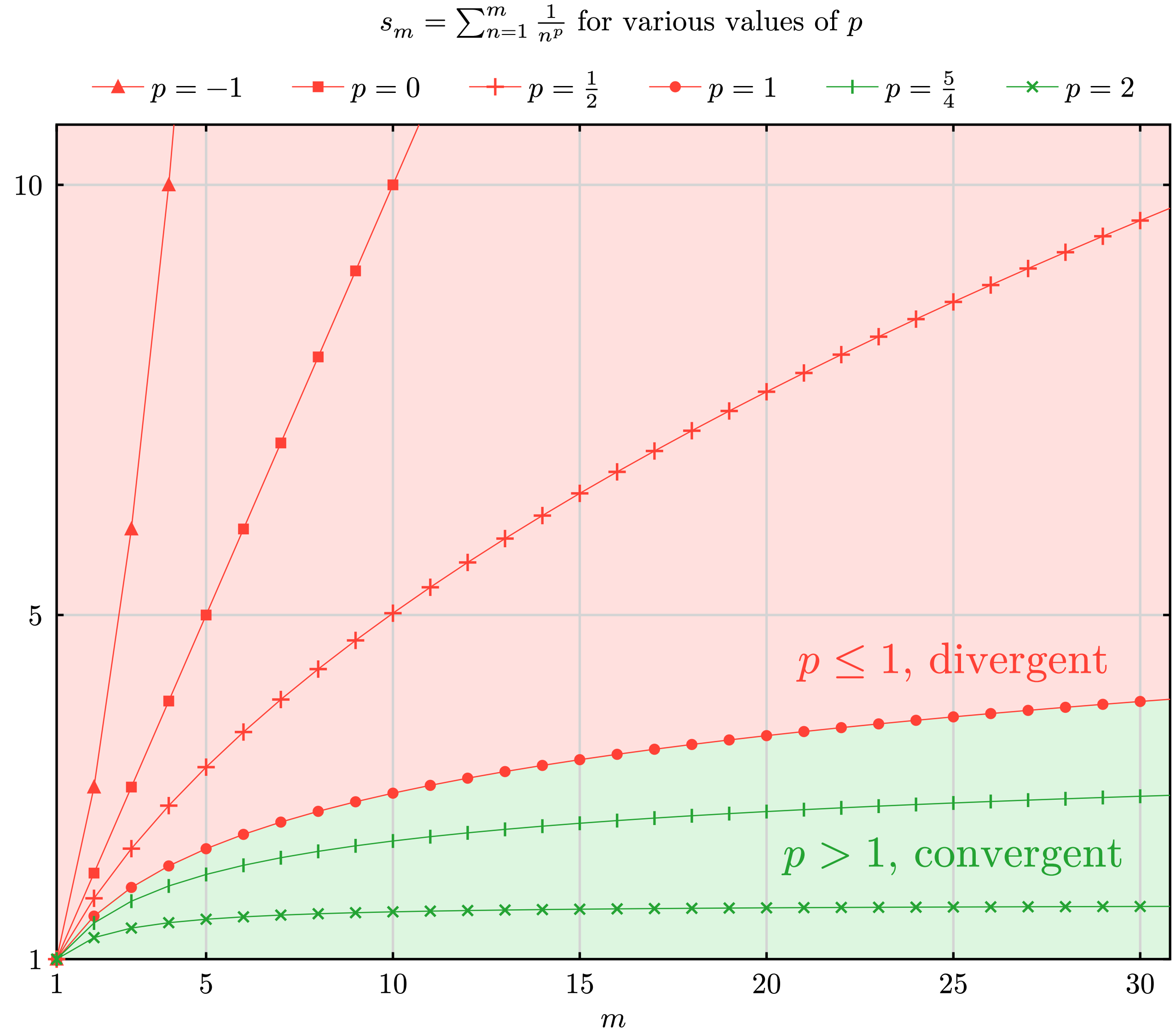
Exercise 2.7.5. Now that we have proved the basis facts about geometric series, supply a proof for Corollary 2.4.7.

Solution. We want to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. If $p \leq 0$ then $\frac{1}{n^p}$ does not converge to zero and it follows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (Theorem 2.7.3). Suppose that $p > 0$ and notice that the sequence $\frac{1}{n^p}$ is positive and decreasing. The Cauchy Condensation Test (Theorem 2.4.6) then implies that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if the series

$$\sum_{n=0}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=0}^{\infty} \left(2^{1-p}\right)^n$$

is convergent. This is a geometric series with common ratio 2^{1-p}, so by Example 2.7.5 this series is convergent if and only if

$$\left|2^{1-p}\right| < 1 \quad \Leftrightarrow \quad 1 - p < 0 \quad \Leftrightarrow \quad p > 1.$$



Exercise 2.7.6. Let's say that a series subverges if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Solution.

(a) This is false. For the bounded sequence $(a_n) = (1, 1, 1, ...)$, the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is $(1, 2, 3, ...)$. This sequence is unbounded and monotone and hence contains no convergent subsequence (Lemma L.7).

(b) This is true. If the sequence of partial sums (s_m) is convergent then any subsequence of (s_n) is convergent; (s_n) itself, for example.

(c) This is true; we will prove the contrapositive statement. Define the sequences of partial sums

$$s_n = |a_1| + \cdots + |a_n| \quad \text{and} \quad t_1 = a_1 + \cdots + a_n.$$

We want to show that if (t_n) has no convergent subsequence, then neither does (s_n). By the Bolzano-Weierstrass Theorem (Theorem 2.5.5) it must be the case that (t_n) is unbounded and, since $t_n \leq s_n$ for all $n \in \mathbf{N}$, it follows that (s_n) is unbounded. Thus (s_n) is an increasing unbounded sequence; such sequences do not have convergent subsequences, as shown in Lemma L.7.

(d) This is false. Consider the sequence $(a_n) = (1, -1, 2, -2, 3, -3, ...)$. The sequence of partial sums is $(s_n) = (1, 0, 2, 0, 3, 0, ...)$, which has the convergent subsequence $(0, 0, 0, ...)$. Thus $\sum_{n=1}^{\infty} a_n$ subverges. However, (a_n) has no convergent subsequence. To see this, observe that for any sequence (x_n) we have

$$(x_n) \text{ has a convergent subsequence} \quad \Rightarrow \quad (|x_n|) \text{ has a convergent subsequence},$$

since if $\lim_{k\to\infty} x_{n_k} = x$ then $\lim_{k\to\infty} |x_{n_k}| = |x|$ by Exercise 2.3.10 (b). Because $(|a_n|) = (1, 1, 2, 2, 3, 3, ...)$ has no convergent subsequence (see Lemma L.7), it follows that (a_n) has no convergent subsequence.

Exercise 2.7.7.

(a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution. The condition that $a_n > 0$ can be relaxed to $a_n \geq 0$ for both parts of this exercise.

(a) Because $na_n \geq 0$ for all $n \in \mathbf{N}$, the Order Limit Theorem (Theorem 2.3.4) and the assumption $l \neq 0$ imply that $l > 0$. Since $na_n \to l$, there exists an $N \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad 0 < \frac{l}{2} < na_n \quad \Rightarrow \quad 0 < \frac{l}{2n} < a_n.$$

Thus the series $\sum_{n=1}^{\infty} a_n$ diverges by comparison (Theorem 2.7.4) with the divergent series $\sum_{n=1}^{\infty} \frac{l}{2n}$ (Example 2.4.5).

(b) Suppose that $\lim(n^2 a_n) = L$; the Order Limit Theorem (Theorem 2.3.4) implies that $L \geq 0$. There is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad 0 \leq n^2 a_n < L + 1 \quad \Rightarrow \quad 0 \leq a_n < \frac{L+1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{L+1}{n^2}$ is convergent (Corollary 2.4.7), the Comparison Test (Theorem 2.7.4) implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.

Exercise 2.7.8. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution.

(a) This is true. Since the series $\sum_{n=1}^{\infty} |a_n|$ converges, we must have $\lim |a_n| = 0$ by Theorem 2.7.3. There is then an $N \in \mathbf{N}$ such that $0 \leq |a_n| \leq 1$ for $n \geq N$; it follows that $0 \leq |a_n|^2 = a_n^2 \leq |a_n|$ for $n \geq N$. We may now apply the Comparison Test (Theorem 2.7.4) to conclude that $\sum_{n=1}^{\infty} a_n^2$ converges absolutely.

(b) This is false. Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$, so that $\lim b_n = 0$. Notice that $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test (Theorem 2.7.7), but $\sum_{n=1}^{\infty} a_n b_n$ is the divergent harmonic series.

(c) This is true; we will prove that

$$\sum_{n=1}^{\infty} |a_n| \text{ diverges} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2 a_n \text{ diverges}$$

by proving the contrapositive statement

$$\sum_{n=1}^{\infty} n^2 a_n \text{ converges} \quad \Rightarrow \quad \sum_{n=1}^{\infty} |a_n| \text{ converges.}$$

By Theorem 2.7.3 we have $\lim(n^2 a_n) = 0$, which implies that $\lim(n^2 |a_n|) = 0$. We may now apply Exercise 2.7.7 (b) to conclude that $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Exercise 2.7.9 (Ratio Test). Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Solution.

(a) Since $\lim\left|\frac{a_{n+1}}{a_n}\right| = r$ and $r' - r > 0$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \Rightarrow \quad \left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < r' - r \quad \Rightarrow \quad \frac{|a_{n+1}|}{|a_n|} < r' \quad \Rightarrow \quad |a_{n+1}| < |a_n| r'.$$

(b) Since $0 < r' < 1$, the geometric series $\sum_{n=0}^{\infty} (r')^n$ converges by Example 2.7.5.

(c) By part (a) we have

$$|a_{N+n}| < |a_{N+n-1}| r' < |a_{N+n-2}| (r')^2 < \cdots < |a_N| (r')^n$$

for any $n \in \mathbf{N}$. It then follows from part (b) and the Comparison Test (Theorem 2.7.4) that the series

$$\sum_{n=0}^{\infty} |a_{N+n}| = \sum_{n=N}^{\infty} |a_n|$$

is convergent. Since a finite number of terms do not affect convergence, we see that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent; the convergence of $\sum_{n=1}^{\infty} a_n$ is then given by Theorem 2.7.6.

Exercise 2.7.10 (Infinite Products). Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

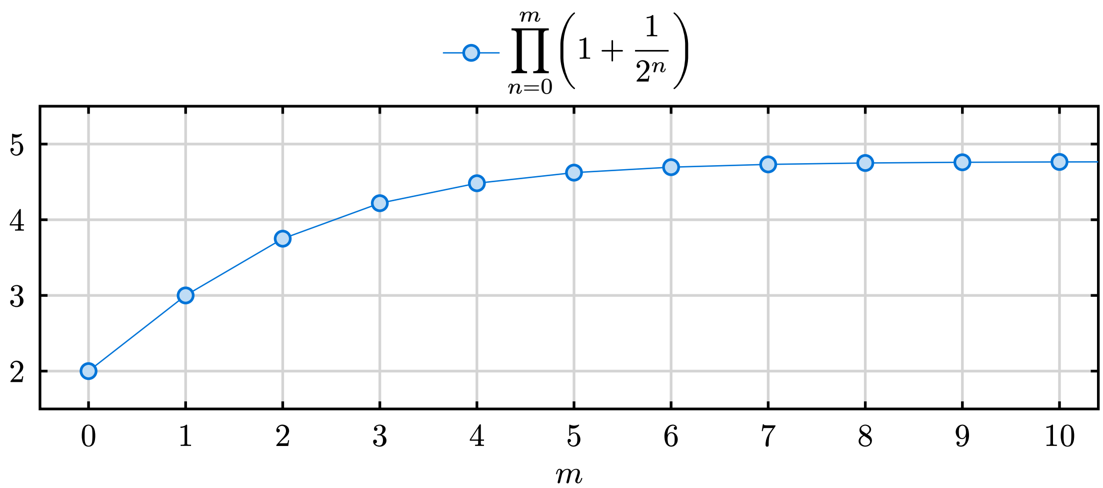
Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

Solution.

(a) This is the infinite product

$$\prod_{n=0}^{\infty} \frac{2^n + 1}{2^n} = \prod_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right).$$

By Exercise 2.4.10 this infinite product converges if and only if the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges. This series is geometric with common ratio $r = \frac{1}{2}$ and hence convergent by Example 2.7.5; it follows that the infinite product converges.



(b) This is the infinite product

$$\prod_{n=1}^{\infty} \frac{2n - 1}{2n} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2n}\right).$$

The sequence of partial products is positive and decreasing, since each term in the partial product satisfies $0 < 1 - \frac{1}{2n} < 1$; the Monotone Convergence Theorem (Theorem 2.4.2) then implies that the infinite product converges.

Indeed, this infinite product converges to zero. To see this, let (p_m) be the sequence of partial products:

$$p_m = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2m-1}{2m}.$$

As noted above, (p_m) is decreasing and satisfies $0 < p_m < 1$ for all $m \in \mathbf{N}$, so we can look at the sequence of reciprocals (p_m^{-1}):

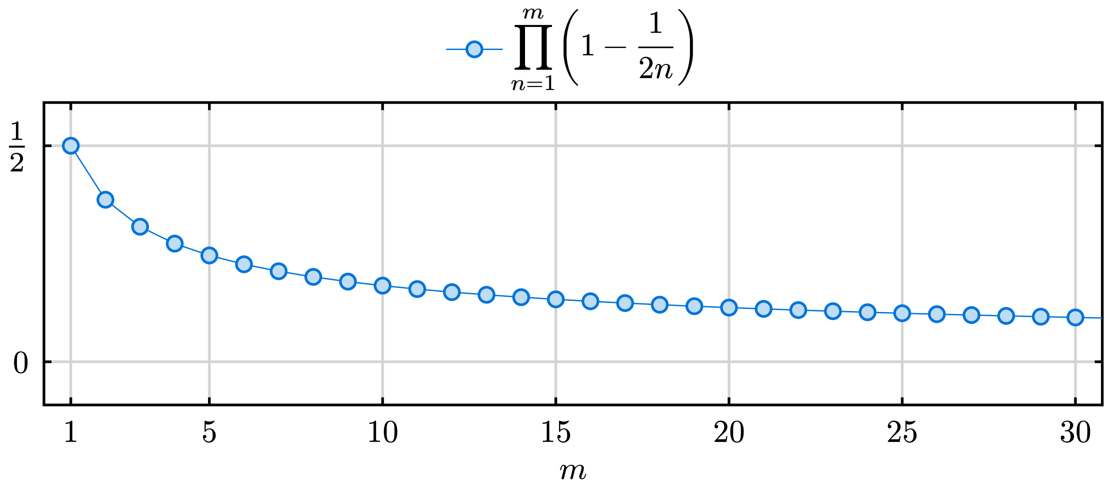
$$\frac{1}{p_m} = \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2m}{2m-1} = \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{2m-1}\right)$$

$$\geq \sum_{n=1}^{m} \frac{1}{2n-1} \geq \frac{1}{2} \sum_{n=1}^{m} \frac{1}{n}.$$

It follows from Example 2.4.5 that (p_m^{-1}) is unbounded above. Thus, for any $\varepsilon > 0$, there is an $M \in \mathbf{N}$ such that $p_M^{-1} > \varepsilon^{-1}$, and since (p_m) is decreasing we then have

$$m \geq M \quad \Rightarrow \quad |p_m| = p_m \leq p_M < \varepsilon.$$

Hence $\lim p_m = 0$.



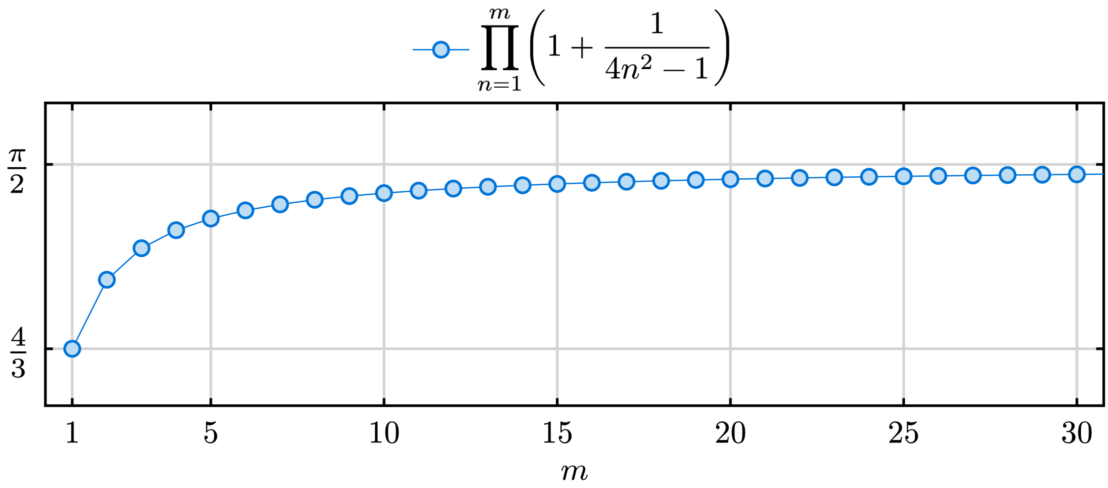
(c) This is the infinite product

$$\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{(2n-1)(2n+1)}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{4n^2-1}\right).$$

By Exercise 2.4.10 this infinite product converges if and only if the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ converges. Observe that for all $n \in \mathbf{N}$ we have

$$n^2 - 1 \geq 0 \quad \Rightarrow \quad 4n^2 - 1 \geq 3n^2 \quad \Rightarrow \quad \frac{1}{4n^2-1} \leq \frac{1}{3n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ is convergent by Corollary 2.4.7, so the Comparison Test (Theorem 2.7.4) implies that the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ is also convergent; it follows that the infinite product $\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\cdots$ converges.



Exercise 2.7.11. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Solution. Consider the series

$$\sum_{n=1}^{\infty} a_n = \underbrace{\frac{1}{1^2}}_{\substack{1 \text{ term} \\ \text{sum} = 1}} + \frac{1}{2^2} + \cdots + \frac{1}{5^2} + \underbrace{\frac{1}{6^2} + \cdots + \frac{1}{6^2}}_{\substack{6^2 \text{ terms} \\ \text{sum} = 1}} + \frac{1}{42^2} + \cdots + \frac{1}{1805^2} + \cdots$$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{1^2} + \underbrace{\frac{1}{2^2} + \cdots + \frac{1}{2^2}}_{\substack{2^2 \text{ terms} \\ \text{sum} = 1}} + \frac{1}{6^2} + \cdots + \frac{1}{41^2} + \underbrace{\frac{1}{42^2} + \cdots + \frac{1}{42^2}}_{\substack{42^2 \text{ terms} \\ \text{sum} = 1}} + \cdots$$

Both (a_n) and (b_n) are strictly positive and decreasing and

$$\sum_{n=1}^{\infty} \min\{a_n, b_n\} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent series. Furthermore, both $\sum a_n$ and $\sum b_n$ diverge since their sequences of partial sums are unbounded: we can find arbitrarily many groupings of consecutive terms which sum to 1, as shown above.

Exercise 2.7.12 (Summation by parts). Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

Solution. For positive integers $n > m$,

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} (s_j - s_{j-1}) y_j \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m-1}^{n-1} s_j y_{j+1} \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_j y_{j+1} + s_n y_{n+1} - s_{m-1} y_m \\
&= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).
\end{aligned}$$

Exercise 2.7.13 (Abel's Test). Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison test to argue that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Solution.

(a) This follows immediately from Exercise 2.7.12, taking $m = 1$ and remembering that $s_0 = 0$.

(b) By assumption the sequence (s_k) is convergent and hence, by Theorem 2.3.2, bounded by some $M > 0$, so that for each $k \in \mathbf{N}$ we have the inequality

$$0 \leq |s_k(y_k - y_{k+1})| = |s_k|(y_k - y_{k+1}) \leq M(y_k - y_{k+1}). \tag{1}$$

Notice that since (y_k) is decreasing and bounded below, the limit $y = \lim_{k \to \infty} y_k$ exists by the Monotone Convergence Theorem (Theorem 2.4.2). It follows that the series $\sum_{k=1}^{\infty} (y_k - y_{k+1})$ is convergent since, letting t_m be the m^{th} partial sum, we have

$$t_m = (y_1 - y_2) + (y_2 - y_3) + \cdots + (y_m - y_{m+1}) = y_1 - y_{m+1} \to y_1 - y \text{ as } m \to \infty.$$

Inequality (1) and the Comparison Test (Theorem 2.7.4) then imply that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ is absolutely convergent and hence convergent (Theorem 2.7.6). From part (a) we have $\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$; it follows that

$$\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty}\left(s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})\right) = y\sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} s_k(y_k - y_{k+1}).$$

Exercise 2.7.14 (Dirichlet's Test). Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Solution.

(a) Abel's Test has the stronger hypothesis that the sequence of partial sums of $\sum_{k=1}^{\infty} x_k$ is convergent (and hence bounded), but the weaker hypothesis that (y_k) need only satisfy $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ without necessarily converging to zero.

Let (s_k) be the k^{th} partial sum of $\sum_{n=1}^{\infty} x_n$; we are given that (s_k) is bounded by some $M > 0$. It follows that

$$0 \leq \left|s_k(y_k - y_{k+1})\right| = |s_k|(y_k - y_{k+1}) \leq M(y_k - y_{k+1}) \tag{1}$$

for each $k \in \mathbf{N}$. The series $\sum_{k=1}^{\infty}(y_k - y_{k+1})$ is convergent since it has m^{th} partial sum

$$(y_1 - y_2) + (y_2 - y_3) + \cdots + (y_m - y_{m+1}) = y_1 - y_{m+1} \to y_1 \text{ as } m \to \infty.$$

Inequality (1) and the Comparison Test (Theorem 2.7.4) then imply that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ is absolutely convergent and hence convergent (Theorem 2.7.6). Since (s_k) is bounded and $\lim y_k = 0$, we have $\lim(s_k y_{k+1}) = 0$ by Exercise 2.3.9 (b). It follows that

$$\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty}\left(s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})\right) = \sum_{k=1}^{\infty} s_k(y_k - y_{k+1}).$$

(b) The Alternating Series Test (Theorem 2.7.7) can be recovered from Dirichlet's Test by taking $x_k = (-1)^{k+1}$; the sequence of partial sums of $\sum_{k=1}^{\infty} x_k$ is $(1, 0, 1, 0, \ldots)$, which is certainly bounded.

2.8. Double Summations and Products of Infinite Series

Exercise 2.8.1. Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

Solution. The array in question is

$$\begin{matrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

That is, $a_{i,j} = 2^{i-j}$ if $j > i$, $a_{i,j} = -1$ if $j = i$, and $a_{i,j} = 0$ if $j < i$. If we let $f(j)$ be the sum of the first row up to the j^{th} column, then using the formula for the partial sums of a geometric series, we find that

$$\begin{aligned} f(j) &= \begin{cases} -1 & \text{if } j = 1, \\ -1 + \frac{1}{2} + \cdots + \frac{1}{2^{j-1}} = -\frac{1}{2^{j-1}} & \text{if } j \geq 2 \end{cases} \\ &= -\frac{1}{2^{j-1}}. \end{aligned}$$

Since subsequent rows are simply the first row shifted along, we see that $s_{1,1} = f(1)$, $s_{2,2} = f(1) + f(2)$, $s_{3,3} = f(1) + f(2) + f(3)$, and in general

$$s_{n,n} = \sum_{j=1}^{n} f(j) = \sum_{j=1}^{n} -\frac{1}{2^{j-1}} = -\sum_{j=0}^{n-1} \frac{1}{2^j}.$$

It follows that

$$\lim_{n\to\infty} s_{n,n} = -\sum_{j=0}^{\infty} \frac{1}{2^j} = -2.$$

At the beginning of Section 2.1, we found that summing along the rows first gave a value of 0 for the double sum, whereas summing down the columns first gave a value of -2.

Exercise 2.8.2. Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some real numbers b_i and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Solution. For each $i \in \mathbf{N}$, Theorem 2.7.6 implies that the series $\sum_{j=1}^{\infty} a_{i,j}$ converges to some real number c_i. Observe that

$$0 \le |c_i| = \left|\sum_{j=1}^{\infty} a_{i,j}\right| \le \sum_{j=1}^{\infty}|a_{i,j}| = b_i.$$

Since $\sum_{i=1}^{\infty} b_i$ converges, the Comparison Test (Theorem 2.7.4) implies that the series $\sum_{i=1}^{\infty} c_i$ is absolutely convergent and hence convergent (Theorem 2.7.6).

Exercise 2.8.3.

(a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

Solution.

(a) Since $|a_{i,j}| \ge 0$ for all positive integers i and j, the sequence $(t_{n,n})$ is increasing and bounded above by the real number $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{i,j}|$. Thus $(t_{n,n})$ converges by the Monotone Convergence Theorem (Theorem 2.4.2).

(b) Suppose $n > m$ are positive integers. By examining the following array,

$$\begin{array}{ccc|ccc}
a_{1,1} & \cdots & a_{1,m} & a_{1,m+1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{m,1} & \cdots & a_{m,m} & a_{m,m+1} & \cdots & a_{m,n} \\
\hline
a_{m+1,1} & \cdots & a_{m+1,m} & a_{m+1,m+1} & \cdots & a_{m+1,n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n,1} & \cdots & a_{n,m} & a_{n,m+1} & \cdots & a_{n,n}
\end{array}$$

we see that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}-\sum_{i=1}^{m}\sum_{j=1}^{m}a_{i,j}=\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{i,j}+\sum_{i=m+1}^{m}\sum_{j=1}^{n}a_{i,j}.$$

In other words, the difference of the entire "square" and the top left "square" is equal to the sum of the top right "square" (in red) and the bottom "rectangle" (in blue).

Let $\varepsilon>0$ be given. Since $(t_{n,n})$ is an increasing Cauchy sequence, there exists an $N\in\mathbf{N}$ such that

$$n>m\geq N\quad\Rightarrow\quad\left|t_{n,n}-t_{m,m}\right|=t_{n,n}-t_{m,m}<\varepsilon.$$

For such n and m, observe that

$$\begin{aligned}\left|s_{n,n}-s_{m,m}\right|&=\left|\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}-\sum_{i=1}^{m}\sum_{j=1}^{m}a_{i,j}\right|\\&=\left|\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{i,j}+\sum_{i=m+1}^{m}\sum_{j=1}^{n}a_{i,j}\right|\\&\leq\sum_{i=1}^{m}\sum_{j=m+1}^{n}\left|a_{i,j}\right|+\sum_{i=m+1}^{m}\sum_{j=1}^{n}\left|a_{i,j}\right|\\&=t_{n,n}-t_{m,m}\\&<\varepsilon.\end{aligned}$$

Thus $(s_{n,n})$ is Cauchy and hence convergent.

Exercise 2.8.4.

(a) Let $\varepsilon>0$ be arbitrary and argue that there exists an $N_1\in\mathbf{N}$ such that $m,n\geq N_1$ implies $B-\frac{\varepsilon}{2}<t_{mn}\leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn}-S|<\varepsilon$$

for all $m,n\geq N$.

Solution.

(a) By Lemma 1.3.8 there exist positive integers k,ℓ such that $B-\frac{\varepsilon}{2}<t_{k,\ell}\leq B$. Let $N_1=\max\{k,\ell\}$. Since each $\left|a_{i,j}\right|$ is positive, $(t_{m,n})$ is increasing in both m and n; it follows that for $m,n\geq N_1$ we have $B-\frac{\varepsilon}{2}<t_{m,n}\leq B$.

(b) Because $\lim_{n\to\infty}s_{n,n}=S$, there is an $N_2\in\mathbf{N}$ such that $\left|s_{n,n}-S\right|<\frac{\varepsilon}{2}$ for all $n\geq N_2$. Let $N=\max\{N_1,N_2\}$ and suppose that $m,n>N$. Similarly to Exercise 2.8.3 (b), we have

$$\begin{aligned}
\left|s_{m,n} - s_{N,N}\right| &= \left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{i,j}\right| \\
&= \left|\sum_{i=1}^{N}\sum_{j=N+1}^{n} a_{i,j} + \sum_{i=N+1}^{m}\sum_{j=1}^{n} a_{i,j}\right| \\
&\leq \sum_{i=1}^{N}\sum_{j=N+1}^{n} \left|a_{i,j}\right| + \sum_{i=N+1}^{m}\sum_{j=1}^{n}\left|a_{i,j}\right| \\
&= t_{m,n} - t_{N,N} \\
&\leq B - t_{N,N} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

It follows that

$$\left|s_{m,n} - S\right| \leq \left|s_{m,n} - s_{N,N}\right| + \left|s_{N,N} - S\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Exercise 2.8.5.

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \varepsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Solution.

(a) For any $n \geq N$, observe that

$$\begin{aligned}
|(r_1 + \cdots + r_m) - S| &\leq \left|(r_1 + \cdots + r_m) - s_{m,n}\right| + \left|s_{m,n} - S\right| \\
&< \left|(r_1 + \cdots + r_m) - \left(\sum_{j=1}^{n} a_{1,j} + \cdots + \sum_{j=1}^{n} a_{m,j}\right)\right| + \varepsilon \\
&\leq \left|r_1 - \sum_{j=1}^{n} a_{1,j}\right| + \cdots + \left|r_m - \sum_{j=1}^{n} a_{m,j}\right| + \varepsilon.
\end{aligned}$$

Since this is true for any $n \geq N$ and for any given i we have $\sum_{j=1}^{\infty} a_{i,j} = r_i$, taking the limit in n on both sides of the inequality

$$|(r_1 + \cdots + r_m) - S| < \left|r_1 - \sum_{j=1}^{n} a_{1,j}\right| + \cdots + \left|r_m - \sum_{j=1}^{n} a_{m,j}\right| + \varepsilon$$

gives us $|(r_1 + \dots + r_m) - S| \le \varepsilon$. Thus $\lim_{m\to\infty} \sum_{i=1}^{m} r_i = S$, i.e. $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = S$.

(b) Fix $j \in \mathbf{N}$ and let (x_n) be the sequence of partial sums of the series $\sum_{i=1}^{\infty}|a_{i,j}|$, i.e.

$$x_n = |a_{1,j}| + |a_{2,j}| + \dots + |a_{n,j}|.$$

Because each $|a_{i,j}|$ is a term of the convergent series $\sum_{j=1}^{\infty}|a_{i,j}| = r_i$, which has only non-negative terms, we see that $|a_{i,j}| \le r_i$, so that

$$x_n \le r_1 + r_2 + \dots + r_n \le \sum_{i=1}^{\infty} r_i,$$

where the last inequality follows since each r_i is non-negative. Thus (x_n) is an increasing and bounded sequence and hence converges by the Monotone Convergence Theorem. It follows that $\sum_{i=1}^{\infty} a_{i,j}$ converges to some (non-negative) real number c_j.

Let $\varepsilon > 0$ be given. As in Exercise 2.8.4, there is an $N \in \mathbf{N}$ such that $|s_{m,n} - S| < \varepsilon$ for all $m, n \ge N$. We can write $s_{m,n}$ as

$$s_{m,n} = \sum_{i=1}^{m} a_{i,1} + \sum_{i=1}^{m} a_{i,2} + \dots + \sum_{i=1}^{m} a_{i,n}.$$

Suppose that $m, n \ge N$ and observe that

$$\begin{aligned}
|(c_1 + \dots + c_n) - S| &\le |(c_1 + \dots + c_n) - s_{m,n}| + |s_{m,n} - S| \\
&< \left|(c_1 + \dots + c_m) - \left(\sum_{i=1}^{m} a_{i,1} + \dots + \sum_{i=1}^{m} a_{i,n}\right)\right| + \varepsilon \\
&\le \left|c_1 - \sum_{i=1}^{m} a_{i,1}\right| + \dots + \left|c_n - \sum_{i=1}^{m} a_{i,n}\right| + \varepsilon.
\end{aligned}$$

Since this is true for any $m \ge N$ and for any given j we have $\sum_{i=1}^{\infty} a_{i,j} = c_j$, taking the limit in m on both sides of the inequality

$$|(c_1 + \dots + c_n) - S| < \left|c_1 - \sum_{i=1}^{m} a_{i,1}\right| + \dots + \left|c_n - \sum_{i=1}^{m} a_{i,n}\right| + \varepsilon$$

gives us $|(c_1 + \dots + c_n) - S| \le \varepsilon$. Thus $\lim_{n\to\infty} \sum_{j=1}^{n} c_j = S$, i.e. $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j} = S$.

Exercise 2.8.6.

(a) Assuming the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{n,n}$.

Solution.

(a) Observe that

$$|d_2| = |a_{1,1}| = \sum_{i=1}^{1}\sum_{j=1}^{1}|a_{i,j}|$$

$$|d_2| + |d_3| = |a_{1,1}| + |a_{1,2} + a_{2,1}| \le \sum_{i=1}^{2}\sum_{j=1}^{2}|a_{i,j}|$$

$$|d_2| + |d_3| + |d_4| = |a_{1,1}| + |a_{1,2} + a_{2,1}| + |a_{1,3} + a_{2,2} + a_{3,1}| \le \sum_{i=1}^{3}\sum_{j=1}^{3}|a_{i,j}|,$$

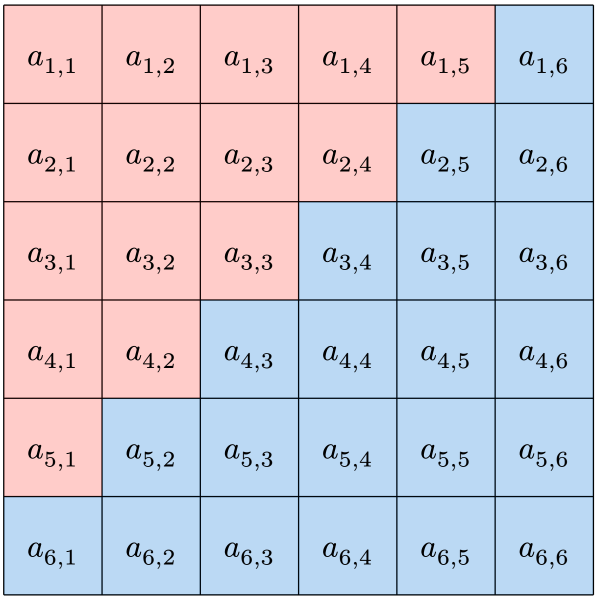
and in general for $n \ge 2$,

$$\sum_{k=2}^{n}|d_k| \le \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}|a_{i,j}| \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{i,j}|.$$

By assumption $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{i,j}|$ is finite, so the sequence $\sum_{k=2}^{n}|d_k|$ is increasing and bounded above and hence converges by the Monotone Convergence Theorem.

(b) By considering the following figure, which shows the special case $n = 6$, we see that for each $n \ge 2$,

$$s_{n,n} - \sum_{k=2}^{n} d_k = \sum_{i=1}^{n}\sum_{j=n+1-i}^{n} a_{i,j}.$$



$$s_{6,6} - \sum_{k=2}^{6} d_k = \sum_{i=1}^{6}\sum_{j=7-i}^{6} a_{i,j}$$

Similarly, letting $e_k = |a_{1,k-1}| + |a_{2,k-2}| + \cdots + |a_{k-1,1}|$ for $k \ge 2$, for each $n \ge 2$ we find that

$$t_{n,n} - \sum_{k=2}^{n} e_k = \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} |a_{i,j}|.$$

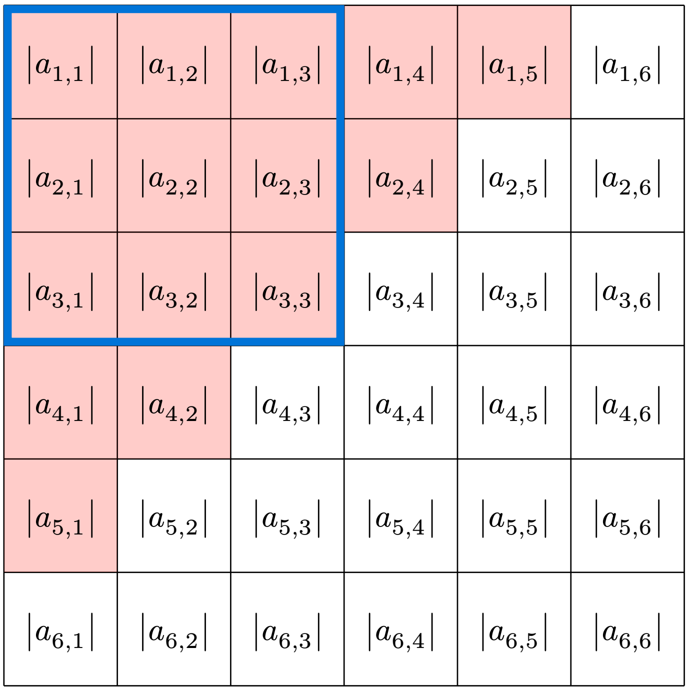
It follows that

$$\left| s_{n,n} - \sum_{k=2}^{n} d_k \right| = \left| \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} a_{i,j} \right| \leq \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} |a_{i,j}| = t_{n,n} - \sum_{k=2}^{n} e_k. \tag{1}$$

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} s_{n,n} = S$ and $(t_{n,n})$ is an increasing Cauchy sequence, there are positive integers N_1, N_2 such that

$$n \geq N_1 \quad \Rightarrow \quad |s_{n,n} - S| < \tfrac{\varepsilon}{2} \quad \text{and} \quad n > m \geq N \quad \Rightarrow \quad t_{n,n} - t_{m,m} < \tfrac{\varepsilon}{2}. \tag{2}$$

Let $N = \max\{N_1, 2N_2\}$ and suppose $n \geq N$. Because $n \geq 2N_2$, each term of t_{N_2,N_2} appears in $\sum_{k=2}^{n} e_k$; see the following figure, which has the special case $n = 6$ and $N_2 = 3$.



$$t_{3,3} \leq \sum_{k=2}^{6} e_k$$

It follows that $t_{N_2,N_2} \leq \sum_{k=2}^{n} e_k$ and thus by (1) and (2) we have

$$\left| s_{n,n} - \sum_{k=2}^{n} d_k \right| \leq t_{n,n} - t_{N_2,N_2} < \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \left| \sum_{k=2}^{n} d_k - S \right| \leq |s_{n,n} - S| + \left| s_{n,n} - \sum_{k=2}^{n} d_k \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We may conclude that $\sum_{k=2}^{\infty} d_k = S$.

Exercise 2.8.7. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

Solution.

(a) Let $\alpha = \sum_{i=1}^{\infty} |a_i|$ and let $\beta = \sum_{j=1}^{\infty} |b_j|$. Notice that for a fixed $i \in \mathbf{N}$ we have

$$\sum_{j=1}^{n} |a_i b_j| = |a_i| \sum_{j=1}^{n} |b_j| \to |a_i|\beta \text{ as } n \to \infty.$$

It follows that

$$\sum_{i=1}^{n} |a_i|\beta = \beta \sum_{i=1}^{n} |a_i| \to \alpha\beta \text{ as } n \to \infty.$$

That is, $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| = \alpha\beta$.

(b) For each $n \in \mathbf{N}$ we have

$$s_{n,n} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{j=1}^{n} b_j \right).$$

The Algebraic Limit Theorem (Theorem 2.3.3) then gives us $\lim_{n\to\infty} s_{n,n} = AB$ and Theorem 2.8.1 then gives the desired result.

Chapter 3. Basic Topology of R

3.2. Open and Closed Sets

Exercise 3.2.1.

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \dots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbf{R}$.

Solution.

(a) This assumption is used when we let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\}$; this minimum is guaranteed to exist because the set $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\}$ is finite (see Lemma L.3). An infinite subset of $\mathbf{R}$ does not necessarily have a minimum. For example, $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ has no minimum.

(b) If we let $O_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbf{N}$, then each O_n is open by Example 3.2.2 (ii), the collection $\{O_1, O_2, O_3, \dots\}$ is countable, and $\bigcap_{n=1}^{\infty} O_n = \{0\} = [0, 0]$, which is non-empty, not equal to $\mathbf{R}$, and closed by Example 3.2.9 (ii).

Exercise 3.2.2. Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \dots\right\} \quad \text{and} \quad B = \{x \in \mathbf{Q} : 0 < x < 1\}.$$

Answer the following questions for each set:

(a) What are the limit points?

(b) Is the set open? Closed?

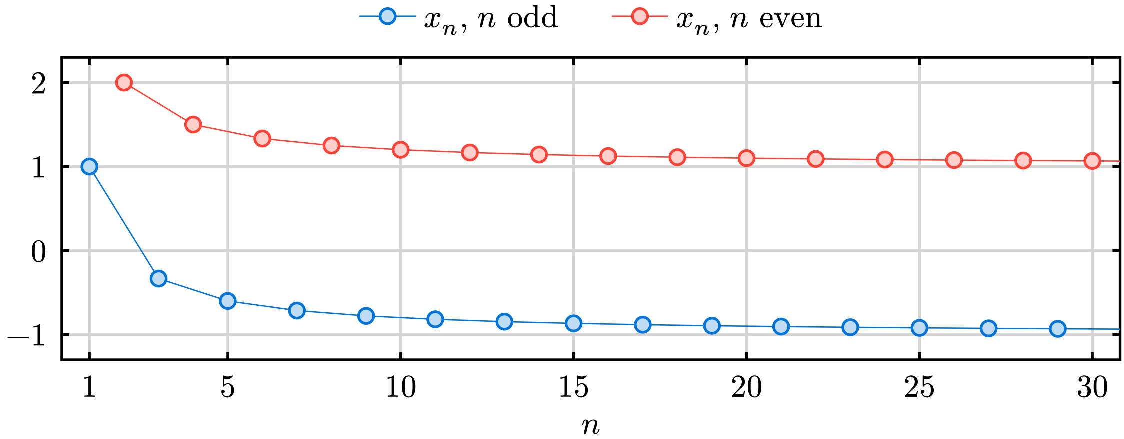
(c) Does the set contain any isolated points?

(d) Find the closure of the set.

Solution. Let us consider the set A first.

(a) Let L_A be the set of limit points of A. We claim that $L_A = \{-1, 1\}$. To see this, first let (x_n) be the sequence given by $x_n = (-1)^n + \frac{2}{n}$ and notice that:
  - $A = \{x_n : n \in \mathbf{N}\}$;
  - $\lim_{n\to\infty} x_{2n-1} = -1$;
  - $x_{2n-1} \neq -1$ for each $n \in \mathbf{N}$;
  - $\lim_{n\to\infty} x_{2n} = 1$;
  - $x_{2n} \neq 1$ for each $n \in \mathbf{N}$.

It follows from Theorem 3.2.5 that -1 and 1 are limit points of A, so that $\{-1, 1\} \subseteq L_A$.



Notice that the blue subsequence is converging to -1 and the red subsequence is converging to 1.

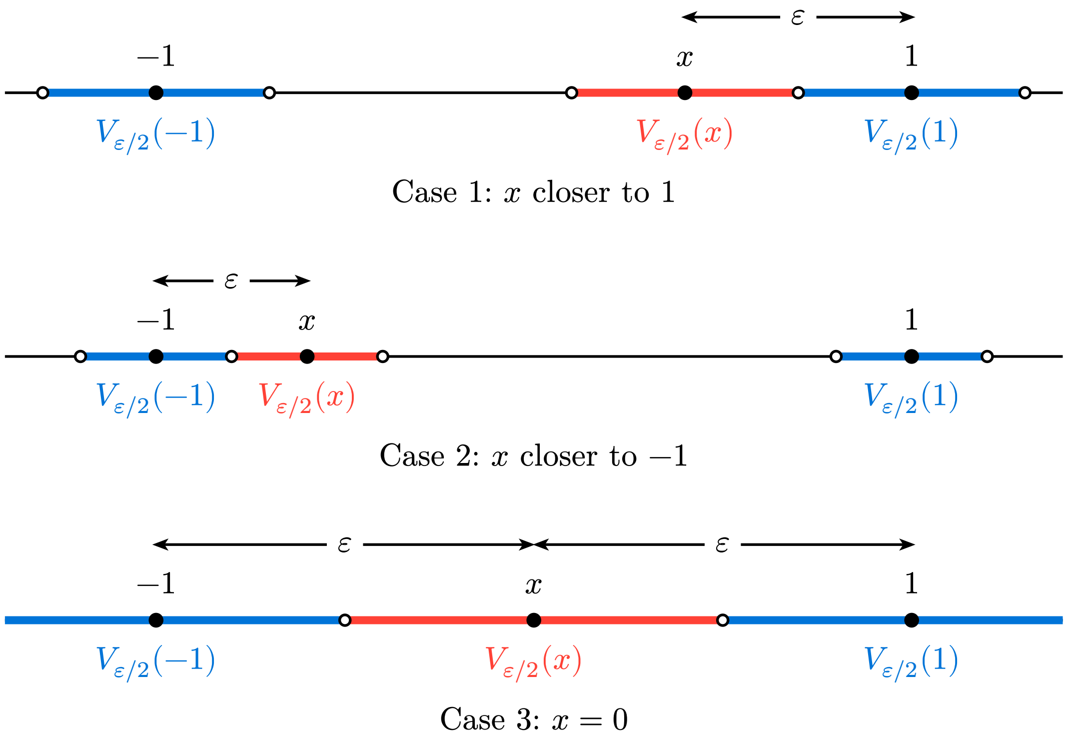
Now suppose that $x \in \mathbf{R}$ is such that $x \notin \{-1, 1\}$. We will show that x is not a limit point of A. Note that the distance from x to each of -1 and 1 is strictly positive, so that

$$\varepsilon = \min\{|x+1|, |x-1|\} > 0.$$

Since $\lim x_{2n-1} = -1$ and $\lim x_{2n} = 1$, the terms of (x_n) (i.e. the elements of A) must eventually be contained inside

$$V_{\varepsilon/2}(-1) \cup V_{\varepsilon/2}(1) = \left(-1 - \tfrac{\varepsilon}{2}, -1 + \tfrac{\varepsilon}{2}\right) \cup \left(1 - \tfrac{\varepsilon}{2}, 1 + \tfrac{\varepsilon}{2}\right).$$

Graphically, the terms of (x_n) are eventually contained in the blue intervals in the following diagram.



Our choice of ε is such that $\left[V_{\varepsilon/2}(-1) \cup V_{\varepsilon/2}(1)\right] \cap V_{\varepsilon/2}(x) = \emptyset$; notice that the red interval does not intersect either of the blue intervals in the diagram above. Thus there can be only finitely many elements of A in $V_{\varepsilon/2}(x)$. It follows that x cannot possibly be the limit of any sequence of elements of A distinct from x, which by Theorem 3.2.5 is to say that x cannot be a limit point of A. We may conclude that $L_A = \{-1, 1\}$.

(b) A is not open. It is straightforward to check that each $a \in A$ satisfies $a \le 2$ and also that $2 \in A$. Thus, for any $\varepsilon > 0$, we have $2 + \frac{\varepsilon}{2} \in V_\varepsilon(2)$ but $2 + \frac{\varepsilon}{2} \notin A$.

A is not closed either since it does not contain the limit point -1: for any $n \in \mathbf{N}$ we have $(-1)^n + \frac{2}{n} > -1$.

(c) Since $L_A = \{-1, 1\}, 1 \in A$, and $-1 \notin A$, every element of A is an isolated point of A.

(d) The closure is

$$\overline{A} = A \cup L_A = \{-1\} \cup \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \dots\right\}.$$

Now let us consider the set B.

(a) Let L_B be the set of limit points of B. We claim that $L_B = [0, 1]$. To see this, first suppose that $x \in [0, 1]$ and let $\varepsilon > 0$ be given. Observe that

$$V_\varepsilon(x) \cap (0, 1) = (\max\{x - \varepsilon, 0\}, \min\{x + \varepsilon, 1\}).$$

This is a proper interval contained in $(0, 1)$ and hence, by the density of $\mathbf{Q}$ in $\mathbf{R}$, contains infinitely many elements of B. It follows that x is a limit point of B and hence that $[0, 1] \subseteq L_B$.

If x is a limit point of B then by Theorem 3.2.5 it must be the case that x is the limit of a sequence of elements of B. The Order Limit Theorem (Theorem 2.3.4) then implies that $0 \le x \le 1$, so that $L_B \subseteq [0, 1]$. We may conclude that $L_B = [0, 1]$.

(b) B is not open since for any $x \in B$ and $\varepsilon > 0$, the neighbourhood $V_\varepsilon(x)$ will contain irrational numbers (Corollary 1.4.4) and hence cannot be contained in B. Neither is B closed, since it does not contain the limit point 0.

(c) B does not have any isolated points, since $B \subseteq [0, 1] = L_B$.

(d) We have $\overline{B} = B \cup L_B = L_B = [0, 1]$.

Exercise 3.2.3. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ε-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbf{Q}$.

(b) $\mathbf{N}$.

(c) $\{x \in \mathbf{R} : x \neq 0\}$.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbf{N}\}$.

Solution.

(a) $\mathbf{Q}$ is not open since $0 \in \mathbf{Q}$ but, by Corollary 1.4.4, there are infinitely many irrational numbers contained in $V_\varepsilon(0)$ for any $\varepsilon > 0$. $\mathbf{Q}$ is not closed either, since Theorem 3.25 and Theorem 3.2.10 show that $\sqrt{2} \notin \mathbf{Q}$ is a limit point of $\mathbf{Q}$.

(b) $\mathbf{N}$ is not open since $1 \in \mathbf{N}$ but $V_\varepsilon(1)$ contains infinitely many non-integers for any $\varepsilon > 0$. However, $\mathbf{N}$ is closed. Observe that

$$\mathbf{N}^{c} = (-\infty, 1) \cup \bigcup_{n=1}^{\infty} (n, n+1),$$

i.e. $\mathbf{N}^{c}$ is a union of open intervals. It follows from Theorem 3.2.3 (i) that $\mathbf{N}^{c}$ is open and hence that $\mathbf{N}$ is closed (Theorem 3.2.13).

(c) Let E be the set in question and notice that $E = (-\infty, 0) \cup (0, \infty)$, a union of two open intervals; it follows that E is open. However, E is not closed: notice that $\frac{1}{n} \in E$ for each $n \in \mathbf{N}$ and $\frac{1}{n} \to 0 \notin E$.

(d) Let E be the set in question and note that each $x \in E$ satisfies $x \geq 1$. It follows that for all $\varepsilon > 0$ we have $1 - \frac{\varepsilon}{2} \in V_\varepsilon(1)$ but $1 - \frac{\varepsilon}{2} \notin E$. Consequently, E is not open. E is not closed either. From Example 2.4.4 we know that $\sum_{k=1}^{\infty} \frac{1}{k^2} = L$ for some $L \in \mathbf{R}$. Observe that for any $n \in \mathbf{N}$,

$$L - \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=n+1}^{\infty} \frac{1}{k^2} > \frac{1}{(n+1)^2} > 0 \quad \Rightarrow \quad L \neq \sum_{k=1}^{n} \frac{1}{k^2}.$$

This implies that L is a limit point of E (Theorem 3.2.5), and also that $L \notin E$. It follows that E is not closed.

(e) Let E be the set in question. As in part (d) we have $1 \in E$ and $x \geq 1$ for all $x \in E$ it follows that E is not open. However, E is closed. Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$, so that $E = \{s_n : n \in \mathbf{N}\}$, and notice that if E had a limit point then Theorem 3.2.5 would imply that the sequence (s_n) contains a convergent subsequence—but (s_n) is an increasing and unbounded sequence and hence contains no convergent subsequences (Lemma L.7). Thus E has no limit points and it follows that E is closed.

Exercise 3.2.4. Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

(b) Can an open set contain its supremum?

Solution.

(a) If $s \in A$ then certainly $s \in \overline{A}$, so suppose that $s \notin A$. For each $n \in \mathbf{N}$ we may use Lemma 1.3.8 to choose some $a_n \in A$ satisfying $s - \frac{1}{n} < a_n < s$ (the last inequality is strict as $s \notin A$). The Squeeze Theorem then implies that $\lim a_n = s$ and thus, by Theorem 3.2.5, s is a limit point of A, whence $s \in \overline{A}$.

(b) An open set cannot contain its supremum. Suppose that $A \subseteq \mathbf{R}$ is open and $x \in A$. There then exists an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq A$, which implies that $x + \frac{\varepsilon}{2} \in A$. It follows that x is not the supremum of A.

Exercise 3.2.5. Prove Theorem 3.2.8.

Solution. Theorem 3.2.8 states that a set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

First suppose that every Cauchy sequence contained in F has a limit that is also an element of F and let $x \in \mathbf{R}$ be a limit point of F. By Theorem 3.2.5 there is a sequence (x_n) contained in F such that $\lim x_n = x$. Because convergent sequences are also Cauchy sequences (Theorem 2.6.4), our hypothesis guarantees that $x \in F$. Thus F contains each of its limit points, i.e. F is closed.

Now suppose that there exists a Cauchy sequence (x_n) contained in F satisfying $x = \lim x_n \notin F$. As (x_n) is entirely contained in F and $x \notin F$, it must be the case that $x_n \neq x$ for each $n \in \mathbf{N}$. It follows from Theorem 3.2.5 that x is a limit point of F. Thus F does not contain each of its limit points, i.e. F is not closed.

Exercise 3.2.6. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbf{R}$.

(b) The Nested Interval Property remains true if the "closed interval" is replaced by "closed set".

(c) Every nonempty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

(e) The Cantor set is closed.

Solution.

(a) This is false: consider the open set $\mathbf{R} \setminus \left\{\sqrt{2}\right\} = \left(-\infty, \sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$.

(b) This is false. Consider the nested closed sets $[n, \infty)$ for $n \in \mathbf{N}$. The Archimedean Property shows that

$$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$

(c) This is true. Suppose that A is open and non-empty, so that there exists some $x \in A$ and some $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq A$. The density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) implies that there is some rational number contained in $V_\varepsilon(x)$ and hence in A.

(d) This is false. Consider the set

$$E = \left\{\sqrt{2}\right\} \cup \left\{\sqrt{2} + \frac{\sqrt{2}}{n} : n \in \mathbf{N}\right\}.$$

This is a bounded infinite set which contains only irrational numbers. Furthermore, an argument similar to the one given in Exercise 3.2.2 (a) shows that $\sqrt{2}$ is the only limit point of E and thus E is closed.

(e) This is true. Because each C_n is the union of 2^n closed intervals, Theorem 3.2.14 (i) shows that each C_n is closed. It follows that $C = \bigcap_{n=1}^{\infty} C_n$ is an intersection of closed sets and hence is itself closed (Theorem 3.2.14 (ii)).

Exercise 3.2.7. Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Solution.

(a) Suppose that $x \in \mathbf{R}$ is a limit point of L; we will show that x is a limit point of A also. Let $\varepsilon > 0$ be given. Because x is a limit point of L, there exists some $y \in L$ such that $0 < |x - y| < \frac{\varepsilon}{2}$, and then since y is a limit point of A, there exists some $a \in A$ such that $|y - a| < |x - y|$. Notice that:

- $|x - a| \leq |x - y| + |y - a| < 2|x - y| < \varepsilon$, so that $a \in V_\varepsilon(x)$;
- $|x - a| \geq |x - y| - |y - a| > 0$, so that $a \neq x$.

Thus x is a limit point of A, i.e. $x \in L$. We may conclude that L is closed.

(b) Let $\varepsilon > 0$ be given. Because x is a limit point of $A \cup L$, the neighbourhood $V_{\varepsilon/2}(x)$ contains some $y \in A \cup L$ such that $y \neq x$. If $y \in A$ then $V_\varepsilon(x)$ contains a point of A other than x, and if $y \in L$ then the argument given in part (a) shows that $V_\varepsilon(x)$ again contains a point of A other than x. It follows that x is a limit point of A. Thus $\overline{A} = A \cup L$ contains all of its limit points and hence is closed.

Exercise 3.2.8. Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A : x \notin B\}$

(c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A^c}$

Solution.

(a) $\overline{A \cup B}$ is definitely closed by Theorem 3.2.12. It may or may not be open. For example, if $A = B = \mathbf{R}$ then $\overline{A \cup B} = \mathbf{R}$ is open. If $A = (0,1)$ and $B = [0,1]$ then $\overline{A \cup B} = [0,1]$ is not open.

(b) Since $A \setminus B = A \cap B^c$ is the intersection of two open sets, $A \setminus B$ is definitely open. It may or may not be closed. For example, if $A = (0,1)$ and $B = [0,1]$ then $A \setminus B = \emptyset$ is closed. If $A = (0,1)$ and $B = [2,3]$, then $A \setminus B = (0,1)$ is not closed.

(c) $A^c \cup B$ is the union of two closed sets and hence is closed. The complement $(A^c \cup B)^c$ is then definitely open. It may or may not be closed. For example, if $A = B = \mathbf{R}$ then $(A^c \cup B)^c = (\emptyset \cup \mathbf{R})^c = \mathbf{R}^c = \emptyset$ is closed. If $A = (0,1)$ and $B = A^c = (-\infty, 0] \cup [1, \infty)$ then

$$(A^c \cup B)^c = (A^c \cup A^c)^c = (A^c)^c = A$$

is not closed.

(d) This is simply the set B, which is given as definitely closed. It may or may not be open: $B = \mathbf{R}$ is closed and open, whereas $B = [0,1]$ is closed but not open.

(e) We claim that $\overline{A}^c$ is a subset of $\overline{A^c}$. To see this, let L_A be the set of limit points of A and let L_{A^c} be the set of limit points of A^c. Notice that

$$\overline{A}^c = (A \cup L_A)^c = A^c \cap L_A^c \quad \text{and} \quad \overline{A^c} = A^c \cup L_{A^c}.$$

Our claim now follows since $\overline{A}^c \subseteq A^c \subseteq \overline{A^c}$. Given this, we have $\overline{A}^c \cap \overline{A^c} = \overline{A}^c$, which is the complement of a closed set and hence is definitely open. It may or may not be closed. For example, if $A = \emptyset$ then $\overline{A}^c = \emptyset^c = \mathbf{R}$ is closed. If $A = (-\infty, 0)$ then $\overline{A}^c = (-\infty, 0]^c = (0, \infty)$ is not closed.

Exercise 3.2.9 (De Morgan's Laws). A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} = \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}} \quad \text{and} \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} = \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}}.$$

(b) Now, provide the details for the proof of Theorem 3.2.14.

Solution.

(a) We have

$$x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} \iff x \notin \bigcup_{\lambda\in\Lambda} E_\lambda \iff x \notin E_\lambda \text{ for all } \lambda \in \Lambda$$
$$\iff x \in E_\lambda^{\mathrm{c}} \text{ for all } \lambda \in \Lambda \iff x \in \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}}$$

The equality $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} = \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}}$ follows. Similarly,

$$x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} \iff x \notin \bigcap_{\lambda\in\Lambda} E_\lambda \iff x \notin E_\lambda \text{ for some } \lambda \in \Lambda$$
$$\iff x \in E_\lambda^{\mathrm{c}} \text{ for some } \lambda \in \Lambda \iff x \in \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}}$$

Thus $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} = \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}}$.

(b) Suppose we have finitely many closed sets $E_1, \ldots, E_n$ and let $E = E_1 \cup \cdots \cup E_n$. It follows from part (a) that

$$E^{\mathrm{c}} = (E_1 \cup \cdots \cup E_n)^{\mathrm{c}} = E_1^{\mathrm{c}} \cap \cdots \cap E_n^{\mathrm{c}}.$$

Each E_k^{c} is open, so Theorem 3.2.3 (ii) implies that E^{c}, which is a finite intersection of open sets, is also open. It then follows from Theorem 3.2.13 that E is closed.

Now suppose that we have an arbitrary collection $\{E_\lambda : \lambda \in \Lambda\}$ of closed sets and let $E = \bigcap_{\lambda\in\Lambda} E_\lambda$. By part (a),

$$E^{\mathrm{c}} = \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathrm{c}} = \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathrm{c}}.$$

Each E_λ^{c} is open, so Theorem 3.2.3 (i) implies that E^{c}, which is an arbitrary union of open sets, is also open. It then follows from Theorem 3.2.13 that E is closed.

Exercise 3.2.10. Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.

(a) A countable set contained in $[0,1]$ with no limit points.

(b) A countable set contained in $[0,1]$ with no isolated points.

(c) A set with an uncountable number of isolated points.

Solution.

(a) This is impossible. Suppose that $E \subseteq [0,1]$ is countable, i.e. there is a bijection $f : \mathbf{N} \to E$. For $n \in \mathbf{N}$, let $x_n = f(n)$. The sequence (x_n) is certainly bounded, so the Bolzano-Weierstrass Theorem implies that there is a convergent subsequence $(x_{n_k}) \to x$ for some $x \in [0,1]$. It then follows from Theorem 3.2.5 that x is a limit point of E. (If $x_{n_k} = x$ for some $k \in \mathbf{N}$, simply remove this term from the sequence, or consider the sequence as starting from $k+1$; there can be at most one such k because f is injective, so this will not affect the convergence of the subsequence.)

(b) This is possible. Consider the countable set $B = (0,1) \cap \mathbf{Q}$ from Exercise 3.2.2. We showed there that B has no isolated points.

(c) This is impossible. Suppose that E is a subset of $\mathbf{R}$ and let A be the set of isolated points of E. If $x \in A$ then there is an $\varepsilon > 0$ such that $V_\varepsilon(x) \cap E = \{x\}$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exist rational numbers p, q such that $x - \varepsilon < p < x < q < x + \varepsilon$. Thus, letting $U_x = (p,q)$, we have $U_x \cap E = \{x\}$. Define $f : A \to B$ by $f(x) = U_x$, where

$$B = \bigcup_{\substack{p,q \in \mathbf{Q}, \\ p<q}} \{(p,q)\}.$$

Theorems 1.5.6 (i), 1.5.7, and 1.5.8 (i), together with Lemma L.5, show that B is a countable set. Assuming that A is uncountable, the function f cannot possibly be injective. Therefore there must exist $x \neq y \in A$ such that $f(x) = f(y)$, i.e. $U_x = U_y$. This implies that

$$\{x\} = U_x \cap E = U_y \cap E = \{y\} \quad \Rightarrow \quad x = y,$$

contradicting $x \neq y$. Thus A cannot be uncountable.

Exercise 3.2.11.

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

Solution.

(a) First, let us prove the following lemma.

Lemma L.9. If A and B are subsets of $\mathbf{R}$ then $x \in \mathbf{R}$ is a limit point of $A \cup B$ if and only if x is a limit point of A or x is a limit point of B.

Proof. Suppose that $x \in \mathbf{R}$ is a limit point of A and let $\varepsilon > 0$ be given. Because x is a limit point of A, there exists some $a \in A$ such that $a \in V_\varepsilon(x)$ and $a \neq x$. Thus there is an element of $A \cup B$ distinct from x and contained in $V_\varepsilon(x)$; it follows that x is a limit point of $A \cup B$. Replacing A with B in the previous argument shows that if x is a limit point of B, then x is a limit point of $A \cup B$.

Now suppose that x is not a limit point of A and not a limit point of B, i.e. there exist positive real numbers ε_1 and ε_2 such that $V_{\varepsilon_1}(x) \cap A \subseteq \{x\}$ and $V_{\varepsilon_2} \cap B \subseteq \{x\}$. If we let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$, then $V_\varepsilon(x) \cap (A \cup B) \subseteq \{x\}$; it follows that x is not a limit point of $A \cup B$. □

Now let us show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. If $x \in \overline{A \cup B}$ then at least one of the following holds:

- $x \in A \cup B$, in which case $x \in \overline{A} \cup \overline{B}$ since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$;
- x is a limit point of $A \cup B$, in which case Lemma L.9 shows that x is a limit point of A or a limit point of B, whence $x \in \overline{A} \cup \overline{B}$.

In either case, $x \in \overline{A} \cup \overline{B}$ and thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

If $x \in \overline{A}$, then at least one of the following holds:

- $x \in A$, in which case $x \in \overline{A \cup B}$ since $A \subseteq A \cup B \subseteq \overline{A \cup B}$;
- x is a limit point of A, in which case Lemma L.9 shows that x is a limit point of $A \cup B$, whence $x \in \overline{A \cup B}$.

Similarly, $x \in \overline{B}$ implies $x \in \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ and we may conclude that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) The result does not extend to the infinite case. For a counterexample, consider the closed sets $A_n = \left[\frac{1}{n}, 1\right]$ for $n \in \mathbf{N}$:

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0,1]} = [0,1] \quad \text{but} \quad \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} A_n = (0,1].$$

Exercise 3.2.12. Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A \text{ and } x < s\}$ and $\{x : x \in A \text{ and } x > s\}$ are uncountable. Show B is nonempty and open.

Solution. Define the sets

$$B_1 = \{x \in \mathbf{R} : (-\infty, x) \cap A \text{ is uncountable}\}, \quad B_2 = \{x \in \mathbf{R} : (x, \infty) \cap A \text{ is uncountable}\}.$$

We claim that B_1 is non-empty. Indeed, suppose that $B_1 = \emptyset$, i.e. for all $x \in \mathbf{R}$ the intersection $(-\infty, x) \cap A$ is either countable or finite, and observe that

$$A = \mathbf{R} \cap A = \left(\bigcup_{n=1}^{\infty} (-\infty, n) \right) \cap A = \bigcup_{n=1}^{\infty} \big((-\infty, n) \cap A \big).$$

This expresses A as a countable union of countable or finite sets; it follows from Theorem 1.5.8 that A is countable or finite. Given that A is uncountable, it must be the case that B_1 is non-empty.

Next we claim that B_1 is open. Let $x \in B_1$ be given, so that $(-\infty, x) \cap A$ is uncountable. Note that for any $y \in \mathbf{R}$ with $y > x$ we must have $y \in B_1$ also, since

$$\big((-\infty, x) \cap A \big) \subseteq \big((-\infty, y) \cap A \big).$$

Given this, we would like to find an $\varepsilon > 0$ such that $x - \varepsilon \in B_1$; it will follow that $(x - \varepsilon, \infty) \subseteq B_1$, so that $V_\varepsilon(x) \subseteq B_1$. Seeking a contradiction, suppose that for every $\varepsilon > 0$ it holds that $x - \varepsilon \notin B_1$. In particular we have $x - \frac{1}{n} \notin B_1$ for each $n \in \mathbf{N}$, so that $\big(-\infty, x - \frac{1}{n}\big) \cap A$ is either countable or finite for each $n \in \mathbf{N}$. Notice that

$$(-\infty, x) \cap A = \bigcup_{n=1}^{\infty} \left(\left(-\infty, x - \frac{1}{n} \right) \cap A \right).$$

It then follows from Theorem 1.5.8 that $(-\infty, x) \cap A$ is countable or finite, contradicting that $x \in B_1$. Thus there must exist an $\varepsilon > 0$ such that $x - \varepsilon \in B_1$, which, as noted above, implies $V_\varepsilon(x) \subseteq B_1$. Thus B_1 is open. Similar arguments show that B_2 is also non-empty and open.

Now let us show that $B_1 \cup B_2 = \mathbf{R}$. If $x \in \mathbf{R}$ is such that $x \notin B_1$ and $x \notin B_2$, i.e. both $(-\infty, x) \cap A$ and $(x, \infty) \cap A$ are either countable or finite, then observe that

$$A = \mathbf{R} \cap A = \big((-\infty, x) \cap A \big) \cup \big(\{x\} \cap A \big) \cup \big((x, \infty) \cap A \big).$$

This expresses A as a union of three countable or finite sets and it follows from Theorem 1.5.8 that A is either countable or finite. Since A is given as uncountable, it must be the case that there is no such $x \in \mathbf{R}$. That is, $B_1 \cup B_2 = \mathbf{R}$.

Observe that $B = B_1 \cap B_2$. To see that B is non-empty, suppose otherwise, so that $B_1^c = B_2$. This demonstrates that B_1 is closed as well as open (Theorem 3.2.13). However, since B_1 is non-empty and not equal to $\mathbf{R}$ (since B_2 is non-empty), and the empty set and $\mathbf{R}$ are the only sets which are both closed and open (see Exercise 3.2.13), this is a contradiction. Thus B is non-empty. Furthermore, B is open since it is the union of two open sets (Theorem 3.2.3 (i)).

Exercise 3.2.13. Prove that the only open sets that are both open and closed are $\mathbf{R}$ and the empty set $\emptyset$.

Solution. It will suffice to show that if $E \subseteq \mathbf{R}$ is non-empty, open, and closed, then $E = \mathbf{R}$. Since $E \neq \emptyset$ there exists some $x \in E$. Let

$$S = \{t \in \mathbf{R} : t \geq x \text{ and } [x, t] \subseteq E\}.$$

Note that S is non-empty since $x \in S$. We claim that S is unbounded above. To see this, suppose otherwise, so that $s = \sup S$ exists. If $s \notin S$ then for any $\varepsilon > 0$ Lemma 1.3.8 shows that there is some $t \in S$ such that $s - \varepsilon < t < s$ (the second inequality is strict because $s \notin S$). Since $t \in S$ implies $t \in E$, and $t \neq s$, we see that for any $\varepsilon > 0$ the intersection $V_\varepsilon(s) \cap E$ contains a point $t \in E$ other than s. That is, s is a limit point of E. Since E is closed it follows that $s \in E$. If $s \in S$ then certainly $s \in E$, so in either case we have $s \in E$.

Because E is open there then exists an $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq E$. This implies that $\left[x, s + \frac{\varepsilon}{2}\right] \subseteq E$, so that $s + \frac{\varepsilon}{2} \in S$, contradicting that s is the supremum of S. Hence S must be unbounded above and it follows that if $t \geq x$ then $t \in E$. A similar argument with the infimum of the set $\{t \in \mathbf{R} : t \leq x \text{ and } [t, x] \subseteq E\}$ shows that if $t \leq x$ then $t \in E$. Thus $E = \mathbf{R}$.

Exercise 3.2.14. A dual notion to the closure of a set is the interior of a set. The *interior* of E is denoted E° and is defined as

$$E^{\circ} = \{x \in E : \text{there exists } V_\varepsilon(x) \subseteq E\}.$$

Results about closures and interiors possess a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^{\circ} = E$.

(b) Show that $\overline{E}^{c} = (E^{c})^{\circ}$, and similarly that $(E^{\circ})^{c} = \overline{E^{c}}$.

Solution.

(a) Let L be the set of limit points of E and observe that $E \cup L = E$ if and only if $L \subseteq E$. This is exactly the statement that $\overline{E} = E$ if and only if E is closed.

Since $E^{\circ} \subseteq E$, it will suffice to show that E is open if and only if $E \subseteq E^{\circ}$. This is clear once we note that $E \subseteq E^{\circ}$ if and only if, for each $x \in E$, there exists an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq E$.

(b) Let L be the set of limit points of E and observe that

$$\begin{aligned}
x \in \overline{E}^{c} \quad &\Leftrightarrow \quad x \in (E \cup L)^{c} \\
&\Leftrightarrow \quad x \in E^{c} \cap L^{c} \\
&\Leftrightarrow \quad x \notin E \text{ and } x \text{ is not a limit point of } E \\
&\Leftrightarrow \quad \text{there exists an } \varepsilon > 0 \text{ such that } V_\varepsilon(x) \cap E = \emptyset \\
&\Leftrightarrow \quad \text{there exists an } \varepsilon > 0 \text{ such that } V_\varepsilon(x) \subseteq E^{c} \\
&\Leftrightarrow \quad x \in (E^{c})^{\circ}.
\end{aligned}$$

Thus $\overline{E}^{\mathrm{c}} = (E^{\mathrm{c}})^{\mathrm{o}}$. Similarly,

$$\begin{aligned}
x \in (E^{\mathrm{o}})^{\mathrm{c}} &\Leftrightarrow x \notin E^{\mathrm{o}} \\
&\Leftrightarrow \text{for all } \varepsilon > 0, V_\varepsilon(x) \nsubseteq E \\
&\Leftrightarrow \text{for all } \varepsilon > 0, V_\varepsilon(x) \cap E^{\mathrm{c}} \neq \emptyset \\
&\Leftrightarrow (\text{for all } \varepsilon > 0)(x \in E^{\mathrm{c}} \text{ or there exists } y \in V_\varepsilon(x) \cap E^{\mathrm{c}} \text{ with } y \neq x) \\
&\Leftrightarrow x \in E^{\mathrm{c}} \text{ or for all } \varepsilon > 0 \text{ there exists } y \in V_\varepsilon(x) \cap E^{\mathrm{c}} \text{ with } y \neq x \\
&\Leftrightarrow x \in E^{\mathrm{c}} \text{ or } x \text{ is a limit point of } E^{\mathrm{c}} \\
&\Leftrightarrow x \in \overline{E^{\mathrm{c}}}.
\end{aligned}$$

Thus $(E^{\mathrm{o}})^{\mathrm{c}} = \overline{E^{\mathrm{c}}}$.

Exercise 3.2.15. A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.

(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set. (We will see in Section 3.5 that $\mathbf{Q}$ is *not* a G_δ set, nor is $\mathbf{I}$ an F_σ set.)

Solution.

(a) Observe that

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right).$$

(b) For any $n \in \mathbf{N}$ the set $\left(a - \frac{1}{n}, b + \frac{1}{n}\right) \setminus \{a\} = \left(a - \frac{1}{n}, a\right) \cup \left(a, b + \frac{1}{n}\right)$ is the union of two open sets and hence is open. Observe that

$$(a, b] = [a, b] \setminus \{a\} = \left(\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)\right) \setminus \{a\} = \bigcap_{n=1}^{\infty} \left(\left(a - \frac{1}{n}, b + \frac{1}{n}\right) \setminus \{a\}\right).$$

Thus $(a, b]$ is a G_δ set. Next, note that for any $n \in \mathbf{N}$ the set $\left[a + \frac{1}{n}, b - \frac{1}{n}\right] \cup \{b\}$ is the union of two closed sets and hence is closed. Note further that

$$(a, b] = \bigcup_{n=1}^{\infty} \left(\left[a + \frac{1}{n}, b - \frac{1}{n}\right] \cup \{b\}\right).$$

Thus $(a, b]$ is an F_σ set.

(c) Observe that

$$\mathbf{Q} = \bigcup_{r \in \mathbf{Q}} \{r\}.$$

Since $\mathbf{Q}$ is countable, this demonstrates that $\mathbf{Q}$ is an F_σ set. De Morgan's Laws (Exercise 3.2.9) imply that the complement of an F_σ set is a G_δ set (and vice versa), so we have also shown that $\mathbf{I}$ is a G_δ set.

3.3. Compact Sets

Exercise 3.3.1. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Solution. K is non-empty and must be bounded by Theorem 3.3.8, so the Axiom of Completeness guarantees that $\sup K$ and $\inf K$ both exist. Theorem 3.3.8 also shows that K is closed and thus, by Exercise 3.2.14, we have $\overline{K} = K$. It then follows from Exercise 3.2.4 that $\sup K \in \overline{K} = K$; a small modification of Exercise 3.2.4 also shows that $\inf K \in \overline{K} = K$.

Exercise 3.3.2. Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbf{N}$.

(b) $\mathbf{Q} \cap [0, 1]$.

(c) The Cantor set.

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

Solution.

(a) $\mathbf{N}$ is not compact. Consider the increasing and unbounded sequence $(1, 2, 3, \ldots)$. As shown in Lemma L.7, such sequences do not have convergent subsequences.

(b) $\mathbf{Q} \cap [0, 1]$ is not compact. Let $x = \frac{\sqrt{2}}{2} \in (0, 1)$. By Theorem 3.2.10 there is a sequence of rational numbers (x_n) converging to x. Because $0 < x < 1$, this sequence must eventually be contained in $(0, 1)$. By removing a finite number of terms from the start of the sequence if necessary, which will not affect convergence, we may assume that the sequence is entirely contained in $\mathbf{Q} \cap [0, 1]$. It follows from Theorem 2.5.2 that every subsequence of (x_n) also converges to x, which does not belong to $\mathbf{Q} \cap [0, 1]$.

(c) The Cantor set C is compact by Theorem 3.3.8: C is closed by Exercise 3.2.6 (e) and bounded since $C \subseteq [0, 1]$.

(d) Let E be the set in question and let $s_n = \sum_{j=1}^{n} \frac{1}{j^2}$. Certainly (s_n) is contained in E and from Example 2.4.4 we know that $\lim s_n = L$ for some $L \in \mathbf{R}$. Exercise 3.2.3 (d) shows that L does not belong to E. Since all subsequences of (s_n) also converge to L (Theorem 2.5.2), it follows that E is not compact.

(e) Let E be the set in question, i.e. $E = \{1\} \cup \left\{1 - \frac{1}{n} : n \in \mathbf{N}\right\}$. Arguing as in Exercise 3.2.2, we see that 1 is the only limit point of E. It follows that E is closed and bounded, and hence compact (Theorem 3.3.8).

Exercise 3.3.3. Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbf{R}$ is closed and bounded, then it is compact.

Solution. Suppose that $K \subseteq \mathbf{R}$ is closed and bounded. If (x_n) is an arbitrary sequence contained in K, then (x_n) must be bounded and so the Bolzano-Weierstrass Theorem implies that there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in \mathbf{R}$. If there exists a $k \in \mathbf{N}$ such that $x_{n_k} = x$ then $x \in K$ since $x_{n_k} \in K$; otherwise $x_{n_k} \neq x$ for all $k \in \mathbf{N}$ and it follows from Theorem 3.2.5 that x is a limit point of K. Thus $x \in K$, since K is closed.

Exercise 3.3.4. Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \setminus F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^c}$

Solution. Throughout this exercise, we will repeatedly use that a subset of $\mathbf{R}$ is compact if and only if it is closed and bounded (Theorem 3.3.8).

(a) K is closed since it is compact, so $K \cap F$ is the intersection of two closed sets and hence is definitely closed (Theorem 3.2.14 (ii)). Certainly the intersection of a bounded set with any other set is again bounded, so since K is bounded by virtue of being compact, we see that $K \cap F$ is bounded as well as closed. It follows that $K \cap F$ is definitely compact.

(b) The closure of any set is closed (Theorem 3.2.12), so $\overline{F^c \cup K^c}$ is definitely closed. However, $\overline{F^c \cup K^c}$ cannot be compact since it is unbounded. To see this, first note that if $E \subseteq \mathbf{R}$ is bounded then E^c must be unbounded, since $\mathbf{R} = E \cup E^c$, the union of two bounded sets is bounded, and $\mathbf{R}$ is not bounded. It follows that K^c is unbounded, since K is bounded as a result of being compact. Because

$$K^c \subseteq F^c \cup K^c \subseteq \overline{F^c \cup K^c},$$

we see that $\overline{F^c \cup K^c}$ must also be unbounded.

(c) Since K is bounded, $K \setminus F$ must also be bounded and thus $K \setminus F$ is compact if and only if it is closed. $K \setminus F$ could be closed: for example, taking $F = \emptyset$. $K \setminus F$ could also fail to be closed. For example, if we take $K = [-2, 2]$ and $F = [-1, 1]$, then $K \setminus F = [-2, 1) \cup (1, 2]$, which is not closed.

(d) First, notice that if $E \subseteq \mathbf{R}$ is bounded by some $M > 0$, i.e. $E \subseteq [-M, M]$, then since $[-M, M]$ is closed it follows from Theorem 3.2.12 that $\overline{E} \subseteq [-M, M]$ also, i.e. $\overline{E}$ is also bounded by M.

Note that since K is bounded, $K \cap F^c = K \setminus F$ must also be bounded. By the previous paragraph, it follows that $\overline{K \cap F^c}$ is bounded. Thus $\overline{K \cap F^c}$ is compact since it is closed and bounded.

Exercise 3.3.5. Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution. Throughout this exercise, we will repeatedly use that a subset of $\mathbf{R}$ is compact if and only if it is closed and bounded (Theorem 3.3.8).

(a) This is true. Suppose we have some collection $\{K_a : a \in A\}$ of compact sets. Each K_a must be closed and bounded and so the intersection $K = \bigcap_{a \in A} K_a$ is also closed (Theorem 3.2.14 (ii)) and bounded. Thus K is compact.

(b) This is false. For each $n \in \mathbf{N}$ let $K_n = [-n, n]$; each K_n is closed and bounded and thus compact. However, $\bigcup_{n=1}^{\infty} K_n = \mathbf{R}$, which is unbounded and hence not compact.

(c) This is false. If we let $A = (0, 1)$ and $K = [0, 1]$, then K is compact since it is closed and bounded but $A \cap K = (0, 1)$, which is not closed and hence not compact.

(d) This is false. See Exercise 3.2.6 (b) for a counterexample.

Exercise 3.3.6. This exercise is meant to illustrate the point made in the opening paragraph to Section 3.3. Verify that the following three statements are true if every blank is filled in with the word "finite". Which are true if every blank is filled in with the word "compact"? Which are true if every blank is filled in with the word "closed"?

(a) Every ______ set has a maximum.

(b) If A and B are ______, then $A + B = \{a + b : a \in A, b \in B\}$ is also ______.

(c) If $\{A_n : n \in \mathbf{N}\}$ is a collection of ______ sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

Solution.

(a) Every non-empty finite set has a maximum by Lemma L.3, and every non-empty compact set has a maximum by Exercise 3.3.1. However, not every closed set has a maximum: $\mathbf{R}$ is closed but has no maximum element.

(b) If A is finite with m elements and B is finite with n elements, then $A+B$ can have at most mn elements since the map $A \times B \to A+B$; $(a,b) \mapsto a+b$ is a surjection. Thus $A+B$ is also finite.

If A and B are compact then so is $A+B$. To see this, let (x_n) be a sequence contained in $A+B$, so that there are sequences (a_n) contained in A and (b_n) contained in B such that $x_n = a_n + b_n$ for each $n \in \mathbf{N}$. Since A is compact, the sequence (a_n) has a subsequence (a_{n_k}) such that $\lim_{k\to\infty} a_{n_k} = a$ for some $a \in A$. Since B is compact, the sequence (b_{n_k}) has a subsequence $\left(b_{n_{k_\ell}}\right)$ such that $\lim_{\ell\to\infty} b_{n_{k_\ell}} = b$ for some $b \in B$. Observe that

$$\lim_{\ell\to\infty} x_{n_{k_\ell}} = \lim_{\ell\to\infty} \left(a_{n_{k_\ell}} + b_{n_{k_\ell}}\right) = \lim_{\ell\to\infty} a_{n_{k_\ell}} + \lim_{\ell\to\infty} b_{n_{k_\ell}} = a + b \in A+B.$$

Thus $A+B$ is compact.

It is not necessarily the case that $A+B$ is closed for closed sets A and B. For a counterexmaple, let $A = \mathbf{N}$ and let $B = \left\{-n + \frac{1}{n} : n \in \mathbf{N}\right\}$. For each $n \in \mathbf{N}$ we have $n + \left(-n + \frac{1}{n}\right) = \frac{1}{n} \in A+B$ it follows from Theorem 3.2.5 that 0 is a limit point of $A+B$. Notice that, for $n, k \in \mathbf{Z}$,

$$n - k + \frac{1}{k} = 0 \quad \Leftrightarrow \quad k = 1 \text{ and } n = 0.$$

Because any element of $A+B$ is of the form $n - k + \frac{1}{k}$ for some positive integers n, k, we see that the limit point 0 fails to belong to $A+B$. Thus $A+B$ is not closed.

(c) Suppose $\{A_n : n \in \mathbf{N}\}$ is a collection of finite sets with the property that every finite subcollection has a non-empty intersection. For each $k \in \mathbf{N}$ let $B_k = \bigcap_{n=1}^{k} A_n$ and notice that each B_k is finite and, by assumption, non-empty. Notice further that

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots.$$

It then follows from Exercise 1.2.3 (b) that the intersection $\bigcap_{n=1}^{\infty} A_n$ is non-empty.

Suppose $\{A_n : n \in \mathbf{N}\}$ is a collection of compact sets with the property that every finite subcollection has a non-empty intersection. For $m \in \mathbf{N}$ define $K_m = \bigcap_{n=1}^{m} A_n$ and observe that each K_m is non-empty by assumption, each K_m is compact by Exercise 3.3.5 (a), and the sequence (K_m) satisfies

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots.$$

It then follows from Theorem 3.3.5 that the intersection $\bigcap_{m=1}^{\infty} K_m = \bigcap_{n=1}^{\infty} A_n$ is non-empty.

The statement is not necessarily true for closed sets. For a counterexample, let $A_n = [n, \infty)$ for $n \in \mathbf{N}$. For a finite subcollection $\{A_{n_1}, \ldots, A_{n_m}\}$ we have

$$\bigcap_{k=1}^{m} A_{n_k} = [N, \infty) \neq \emptyset, \quad \text{where} \quad N = \max_{1 \le k \le m} n_k.$$

However, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Exercise 3.3.7. As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. (Keep in mind that C has zero length and contains no intervals.)

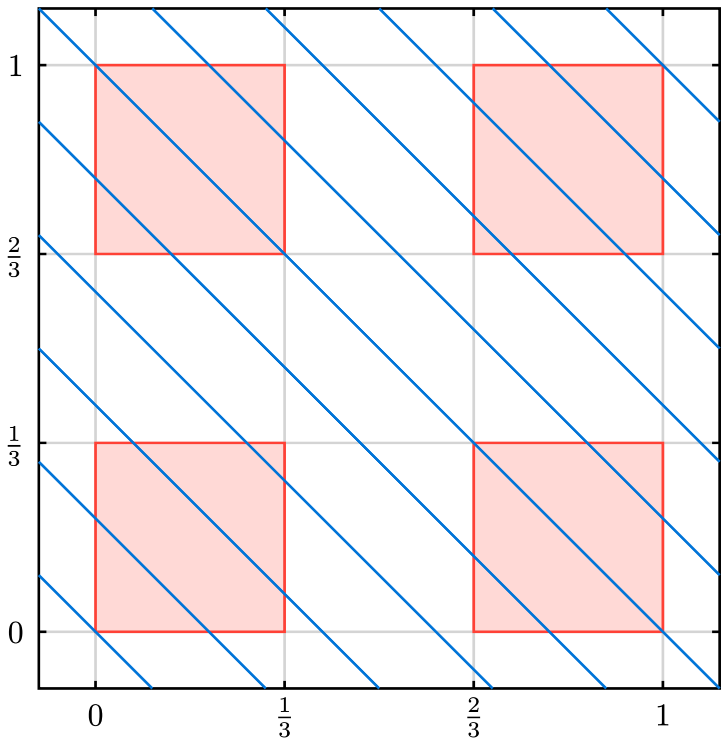
Because $C \subseteq [0, 1], C + C \subseteq [0, 2]$, so we only need to prove the reverse inclusion $[0, 2] \subseteq \{x + y : x, y \in C\}$. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$.

(a) Show that there exist $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for an arbitrary $n \in \mathbf{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired x and y in C satisfying $x + y = s$.

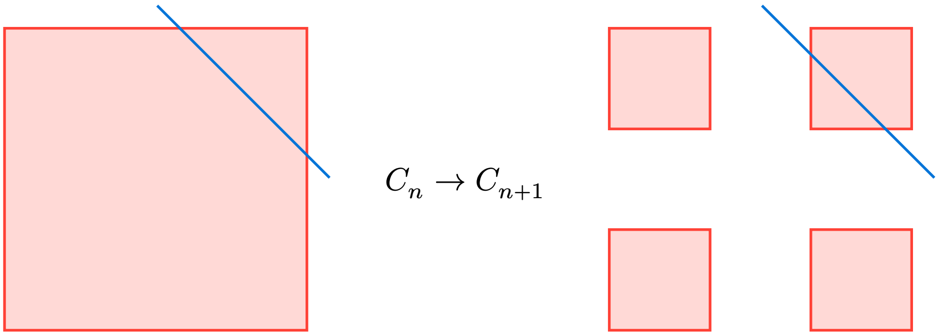
Solution.

(a) If $\frac{s}{2} \in C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ then take $x_1 = y_1 = \frac{s}{2}$, and if $\frac{s}{2} \in \left(\frac{1}{3}, \frac{2}{3}\right)$ then take $x_1 = \frac{s}{2} - \frac{1}{3}$ and $y_1 = \frac{s}{2} + \frac{1}{3}$. In either case we have $x_1, y_1 \in C_1$ and $x_1 + y_1 = s$. Geometrically, we have shown that for any $s \in [0, 2]$, the line given by $x + y = s$ must intersect the set $C_1 \times C_1 \subseteq C_0 \times C_0 = [0, 1]^2$.



$C_1 \times C_1$ and $x + y = s$ for various values of $s \in [0, 2]$

We will proceed by induction to show that for any $n \in \mathbf{N}$ we can find $x_n, y_n \in C_n$ such that $x_n + y_n = s$. The base case $n = 1$ was handled above, so suppose that for some $n \in \mathbf{N}$ we have $x_n, y_n \in C_n$ such that $x_n + y_n = s$. Since C_n consists of 2^n closed intervals each of length 3^{-n}, the set $C_n \times C_n$ consists of $(2^n)^2$ closed squares each with side length 3^{-n}. Geometrically, the induction hypothesis guarantees that the line $x + y = s$ intersects the set $C_n \times C_n$ and thus must intersect one of the $(2^n)^2$ closed squares. Moving from C_n to C_{n+1}, the middle third of each of the 2^n intervals is removed. This has the effect of splitting each of the $(2^n)^2$ squares of $C_n \times C_n$ into four subsquares. $C_{n+1} \times C_{n+1}$ then consists of the collection of these subsquares. Now we make the observation that this situation is essentially the same as in the base case: given that the line $x + y = s$ intersects one of the squares of $C_n \times C_n$, it must intersect at least one of the four subsquares after we remove the middle third of the sides of the square. We are then guaranteed the existence of some $x_{n+1}, y_{n+1} \in C_{n+1}$ such that $x_{n+1} + y_{n+1} = s$. This completes the induction step.



Subsquares of $C_n \times C_n$ and $C_{n+1} \times C_{n+1}$ intersecting the line $x + y = s$

(b) The sequence (x_n) is certainly bounded, so by the Bolzano-Weierstrass Theorem there is a convergent subsequence $(x_{n_k}) \to x$ for some $x \in \mathbf{R}$. Similarly, the sequence (y_{n_k}) is bounded and hence has a convergent subsequence $\left(y_{n_{k_\ell}}\right) \to y$ for some $y \in \mathbf{R}$. Because the sequence (C_n) is nested we have $x_{n_{k_\ell}} \in C_1$ for all $\ell \in \mathbf{N}$; it follows that $x \in C_1$ since C_1 is closed. The terms $x_{n_{k_\ell}}$ belong to C_2 provided $n_{k_\ell} \geq 2$, i.e. all but a finite number of terms of $\left(x_{n_{k_\ell}}\right)$ belong to C_2. Since C_2 is closed it must then be the case that $x \in C_2$. Continuing in this fashion, we see that $x \in C_n$ for all $n \in \mathbf{N}$, i.e. $x \in C$. Similarly we obtain $y \in C$. Now observe that,

$$\lim_{\ell \to \infty} \left(x_{n_{k_\ell}} + y_{n_{k_\ell}}\right) = x + y \quad \text{and} \quad \lim_{\ell \to \infty} \left(x_{n_{k_\ell}} + y_{n_{k_\ell}}\right) = \lim_{\ell \to \infty} s = s.$$

Since limits are unique (Theorem 2.2.7), we may conclude that $x + y = s$.

Exercise 3.3.8. Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}.$$

This turns out to be a reasonable definition for the *distance* between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

Solution.

(a) Let $E = \{|x - y| : x \in K \text{ and } y \in L\}$ and notice that E is non-empty (since K and L are non-empty) and bounded below by 0; it follows that $d = \inf E$ exists. By Exercise 1.3.1 (b), for each $n \in \mathbf{N}$ there exist elements $x_n \in K$ and $y_n \in L$ such that

$$d \leq |x_n - y_n| < d + \frac{1}{n}. \tag{1}$$

Since (x_n) is entirely contained in the compact set K, we are guaranteed the existence of a convergent subsequence $(x_{n_k}) \to x_0$ for some $x_0 \in K$. Similarly, because the sequence (y_{n_k}) is entirely contained in the compact set L, there exists a convergent subsequence $(y_{n_{k_\ell}}) \to y_0$ for some $y_0 \in L$. We then have, by Theorem 2.5.2,

$$\lim_{\ell \to \infty} \left| x_{n_{k_\ell}} - y_{n_{k_\ell}} \right| = |x_0 - y_0|.$$

However, inequality (1) and the Squeeze Theorem imply that

$$\lim_{\ell \to \infty} \left| x_{n_{k_\ell}} - y_{n_{k_\ell}} \right| = d.$$

It follows from the uniqueness of limits (Theorem 2.2.7) that $|x_0 - y_0| = d$. Since K and L are disjoint, it must be the case that $x_0 \neq y_0$ and thus $d > 0$.

(b) Let $K = \mathbf{N}$ and $L = \left\{n + \frac{1}{n} : n \geq 2\right\}$ and note that K and L are non-empty and disjoint. Note further that

$$K^{\mathrm{c}} = (-\infty, 1) \cup \bigcup_{n=1}^{\infty} (n, n+1) \quad \text{and} \quad L^{\mathrm{c}} = \left(-\infty, \frac{5}{2}\right) \cup \bigcup_{n=2}^{\infty} \left(n + \frac{1}{n}, n + 1 + \frac{1}{n+1}\right).$$

It follows that K^{c} and L^{c} are both open (Theorem 3.2.3 (i)) and hence that K and L are both closed (Theorem 3.2.13). Letting $E = \{|x - y| : x \in K \text{ and } y \in L\}$ again, note that for each $n \geq 2$, by taking $n \in K$ and $n + \frac{1}{n} \in L$, we have $\frac{1}{n} \in E$. It follows that $d = \inf E = 0$.

Exercise 3.3.9. Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Solution.

(a) Let us proceed by induction. For the base case, $I_0 \cap K = K$ cannot be covered by any finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$ and we have $|I_0| = 2^0 |I_0|$.

Suppose that after n steps we have chosen nested closed intervals $I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{n-1}$ such that, for each $0 \le m \le n-1$, $I_m \cap K$ cannot be covered by any finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$ and $|I_m| = 2^{-m}|I_0|$. Suppose that $I_{n-1} = [a, c]$ and let $b = \frac{a+c}{2}$. Note that if both of the sets $[a, b] \cap K$ and $[b, c] \cap K$ could be covered by a finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$, then $I_{n-1} \cap K$ could also be finitely covered. By assumption this is not the case, so at least one of the intervals $[a, b]$ or $[b, c]$ must have the property that its intersection with K cannot be finitely covered. Let I_n be this interval and note that $I_n \subseteq I_{n-1}$. Furthermore, since $|I_{n-1}| = 2^{-n+1}|I_0|$ and $|I_n| = \frac{1}{2}|I_{n-1}|$, we have $|I_n| = 2^{-n}|I_0|$. This completes the induction step and thus we obtain the desired sequence of nested closed intervals.

(b) For each $n \in \mathbf{N}$, $I_n \cap K$ is the intersection of two compact sets and hence is itself compact (Exercise 3.3.5 (a)). Furthermore, since the sequence (I_n) is nested, the sequence $(I_n \cap K)$ is also nested. It follows from Theorem 3.3.5 that there exist some $x \in \bigcap_{n=1}^\infty (I_n \cap K) = K \cap \bigcap_{n=1}^\infty I_n$.

(c) Because x belongs to the open set O_{λ_0}, there exists an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq O_{\lambda_0}$, and since $\lim |I_n| = 0$ there exists an $N \in \mathbf{N}$ such that $|I_N| < \frac{\varepsilon}{2}$. Thus, since $x \in I_N$, we must have $I_N \subseteq V_\varepsilon(x)$ and hence $(I_N \cap K) \subseteq V_\varepsilon(x)$. This implies that $(I_N \cap K) \subseteq O_{\lambda_0}$, contradicting the fact that $I_N \cap K$ cannot be covered by any finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$.

Exercise 3.3.10. Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine-Borel Theorem.

Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

Solution.

(a) Since $a \in [a, b]$ there must be some O_{λ_0} such that $a \in O_{\lambda_0}$, so that $[a, a]$ is finitely covered; it follows that $a \in S$. Evidently, S is bounded above by b. Thus $s = \sup S$ exists.

(b) Seeking a contradiction, suppose that $s < b$, so that $\varepsilon_1 := \frac{b-s}{2} > 0$. Since $s \in [a, b]$, there exists some O_{λ_0} such that $s \in O_{\lambda_0}$ and thus there is an $\varepsilon_2 > 0$ such that $V_{\varepsilon_2}(s) \subseteq O_{\lambda_0}$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$. By Lemma 1.3.8 there exists an $x \in S$ such that $s - \varepsilon < x \leq s$, so that $x \in V_\varepsilon(s)$ and

$$[a, x] \subseteq O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}$$

for some finite subcollection $\{O_{\lambda_1}, \ldots, O_{\lambda_n}\}$. Observe that $s + \frac{\varepsilon}{2} \leq s + \frac{\varepsilon_1}{2} = \frac{s+b}{2} \in [a, b]$ and

$$\left[a, s + \frac{\varepsilon}{2}\right] \subseteq V_\varepsilon(s) \cup [a, x] \subseteq V_{\varepsilon_2} \cup [a, x] \subseteq O_{\lambda_0} \cup O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}.$$

It follows that $s + \frac{\varepsilon}{2} \in S$, contradicting that s is the supremum of S. Hence it must be the case that $s = b$.

This implies that $[a, b]$ has a finite subcover: since $b \in [a, b]$ there must be some O_{λ_0} such that $b \in O_{\lambda_0}$ and hence some $\varepsilon > 0$ such that $V_\varepsilon(b) \subseteq O_{\lambda_0}$, and since $\sup S = b$ there is some $x \in S$ such that $b - \varepsilon < x \leq b$ and

$$[a, x] \subseteq O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}$$

for some finite subcollection $\{O_{\lambda_1}, \ldots, O_{\lambda_n}\}$. It follows that

$$[a, b] \subseteq V_\varepsilon(b) \cup [a, x] \subseteq O_{\lambda_0} \cup O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}.$$

(c) Let $\{O_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of K. Since K is bounded, it is contained in some closed interval $[a, b]$. Note that since K is closed, the collection $\{K^c\} \cup \{O_\lambda : \lambda \in \Lambda\}$ is an open cover of $\mathbf{R}$ and hence of $[a, b]$; by part (b), there then exists a finite subcover of $[a, b]$. Since K is contained in $[a, b]$, this finite subcover must also cover K, and since $\{K^c\}$ evidently does not cover K, this finite subcover must contain some sets $O_{\lambda_1}, \ldots, O_{\lambda_n}$. It follows that K is covered by the finite collection $\{O_{\lambda_1}, \ldots, O_{\lambda_n}\}$.

Exercise 3.3.11. Consider each of the sets listed in Exercise 3.3.2. For each one that is not compact, find an open cover for which there is no finite subcover.

Solution. The sets from Exercise 3.3.2 which are not compact are $\mathbf{N}, \mathbf{Q} \cap [0,1]$, and

$$E = \left\{ \sum_{k=1}^{n} \frac{1}{k^2} : n \in \mathbf{N} \right\}.$$

Let us consider $\mathbf{N}$ first. For each $n \in \mathbf{N}$, let $O_n = (n-1, n+1)$, so that the collection $\{O_n : n \in \mathbf{N}\}$ covers $\mathbf{N}$. Since each $n \in \mathbf{N}$ belongs to exactly the set O_n and no others, there are in fact no proper subcovers, finite or otherwise.

Next, consider $\mathbf{Q} \cap [0,1]$. Let y be the irrational number $\frac{\sqrt{2}}{2} \in (0,1)$. For each $n \in \mathbf{N}$, define

$$O_n = \left(-\infty, y - \frac{1}{n}\right) \cup \left(y + \frac{1}{n}, \infty\right)$$

and notice that $\bigcup_{n=1}^{\infty} O_n = \mathbf{R} \setminus \{y\}$; it follows that the collection $\{O_n : n \in \mathbf{N}\}$ covers $\mathbf{Q} \cap [0,1]$ since y is irrational. We claim that there can be no finite subcover. If $\{O_{n_1}, ..., O_{n_m}\}$ is some finite subcollection, then let $N = \max\{n_1, ..., n_m\}$ and observe that

$$\bigcup_{k=1}^{m} O_{n_k} = \left(-\infty, y - \frac{1}{N}\right) \cup \left(y + \frac{1}{N}, \infty\right).$$

Notice that

$$\left[y - \frac{1}{N}, y + \frac{1}{N}\right] \cap [0,1] = \left[\max\left\{0, y - \frac{1}{N}\right\}, \min\left\{1, y + \frac{1}{N}\right\}\right].$$

Because this is a proper interval, we are guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$ the existence of a rational number $p \in \left[y - \frac{1}{N}, y + \frac{1}{N}\right] \cap [0,1]$. It follows that $\mathbf{Q} \cap [0,1] \not\subseteq \bigcup_{k=1}^{m} O_{n_k}$.

Now consider the set $E = \{s_n : n \in \mathbf{N}\}$, where $\sum_{k=1}^{n} \frac{1}{k^2}$. We know by the Monotone Convergence Theorem that $L := \lim s_n$ is the supremum of E. Furthermore, as noted in Exercise 3.2.2, L does not belong to E. For each $n \in \mathbf{N}$, let $O_n = \left(-\infty, L - \frac{1}{n}\right)$ and note that

$$\bigcup_{n=1}^{\infty} O_n = (-\infty, L).$$

This must cover E since L is the supremum of E but does not belong to E. We claim that there cannot exist a finite subcover. If $\{O_{n_1}, ..., O_{n_m}\}$ is some finite subcollection, then let $N = \max\{n_1, ..., n_m\}$ and observe that

$$\bigcup_{k=1}^{m} O_{n_k} = \left(-\infty, L - \frac{1}{N}\right).$$

Since $\lim s_n = L$, the sequence (s_n) must eventually be contained in the interval $\left(L - \frac{1}{N}, L + \frac{1}{N}\right)$ and it follows that $\{O_{n_1}, ..., O_{n_m}\}$ cannot cover E.

Exercise 3.3.12. Using the concept of open covers (and explicitly avoiding the Bolzano-Weierstrass Theorem), prove that every bounded infinite set has a limit point.

Solution. We will prove the contrapositive statement. That is, if $E \subseteq \mathbf{R}$ is bounded, then

$$E \text{ has no limit points} \quad \Rightarrow \quad E \text{ is finite.}$$

If E is empty we are done. Otherwise, each $x \in E$ must be an isolated point, i.e. there exists some $\varepsilon_x > 0$ such that $V_{\varepsilon_x}(x) \cap E = \{x\}$. Notice that the collection $\{V_{\varepsilon_x}(x) : x \in E\}$ is an open cover of E. Since E has no limit points, E must be closed; the Heine-Borel Theorem (Theorem 3.3.8) then implies that there exist finitely many points $\{x_1, ..., x_n\}$ such that

$$E \subseteq V_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup V_{\varepsilon_{x_n}}(x_n).$$

This implies that

$$E = E \cap \left(V_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup V_{\varepsilon_{x_n}}(x_n)\right) = \left(V_{\varepsilon_{x_1}}(x_1) \cap E\right) \cup \cdots \cup \left(V_{\varepsilon_{x_n}}(x_n) \cap E\right)$$
$$= \{x_1\} \cup \cdots \cup \{x_n\} = \{x_1, ..., x_n\}.$$

Thus E is finite.

Exercise 3.3.13. Let's call a set *clompact* if it has the property that every *closed* cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbf{R}$.

Solution. Let E be a subset of $\mathbf{R}$. Suppose that E is finite. If E is empty then certainly E is clompact, so suppose that $E = \{x_1, ..., x_n\}$ and let $\{F_\lambda : \lambda \in \Lambda\}$ be a closed cover of E. For each $x_k \in E$, there is some F_{λ_k} such that $x_k \in F_{\lambda_k}$; it follows that $\{F_{\lambda_1}, ..., F_{\lambda_n}\}$ is a finite subcover of E. Thus E is clompact.

Now suppose that E is infinite and consider the closed cover $\{\{x\} : x \in E\}$. Since E is infinite, finitely many singletons cannot possibly cover E. So we have found a closed cover of E which does not admit a finite subcover and thus E is not clompact.

To conclude, the clompact subsets of $\mathbf{R}$ are precisely the finite subsets of $\mathbf{R}$.

3.4. Perfect Sets and Connected Sets

Exercise 3.4.1. If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution. P is closed so $P \cap K$ must be compact by Exercise 3.3.4 (a). However, $P \cap K$ need not be perfect. For a counterexample, consider $P = [0, 1]$ and $K = \{0\}$.

Exercise 3.4.2. Does there exist a perfect set consisting of only rational numbers?

Solution. No. By Theorem 3.4.3 a non-empty perfect set must be uncountable, but any subset of $\mathbf{Q}$ is either finite or countably infinite (Theorem 1.5.6 (i) and Theorem 1.5.7). (Strictly speaking, the empty set is both perfect and a subset of the rationals; I suspect this is not what Abbott had in mind.)

Exercise 3.4.3. Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument.

(a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

(b) Finish the proof by showing that for each $n \in \mathbf{N}$, there exists $c_n \in C \cap C_n$, different from x satisfying $|x - x_n| \leq 1/3^n$.

Solution.

(a) Recall that $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. The endpoints of these intervals are never removed at any subsequent stage of the construction of the Cantor set, so they belong to C. Since $x \in C_1$, it must belong to one of these intervals, say the interval $\left[0, \frac{1}{3}\right]$. If $0 \leq x < \frac{1}{3}$ then take $x_1 = \frac{1}{3}$, and if $x = \frac{1}{3}$ then take $x_1 = 0$. We can make similar choices if $x \in \left[\frac{2}{3}, 1\right]$. In any case, we have chosen an $x_1 \in C \cap C_1$ satisfying $x_1 \neq x$ and $|x - x_1| \leq \frac{1}{3}$.

(b) Let $n \in \mathbf{N}$ be given. The set C_n consists of 2^n disjoint closed intervals each of length 3^{-n}. The endpoints of these intervals are never removed at any subsequent stage of the construction of the Cantor set and thus they belong to C. Since $x \in C$, we have $x \in C_n$ and hence x must belong to one of the disjoint closed intervals of which C_n is composed, say $I = [a, b]$ where $b - a = 3^{-n}$. If $a \leq x < b$ then let $x_n = b$ and if $x = b$ then let $x_n = a$. In either case, we have chosen an $x_n \in C \cap C_n$ satisfying $x \neq x_n$ and $|x - x_n| \leq b - a = 3^{-n}$.

It follows from the Squeeze Theorem that $\lim x_n = x$. Thus x is the limit of a sequence (x_n) contained in C such that $x_n \neq x$ for each $n \in \mathbf{N}$, i.e. x is a limit point of C (Theorem 3.2.5). Hence C contains no isolated points.

Exercise 3.4.4. Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle *fourth* from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Solution. We begin with $B_0 := [0, 1]$ and remove the open middle fourth to obtain $B_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$. Notice that each interval has length $\frac{3}{8}$. Next we remove the open middle fourth from each of the two intervals of B_1 to obtain

$$B_2 = \left(\left[0, \frac{9}{64}\right] \cup \left[\frac{15}{64}, \frac{24}{64}\right]\right) \cup \left(\left[\frac{40}{64}, \frac{49}{64}\right] \cup \left[\frac{55}{64}, 1\right]\right).$$

Notice that each interval has length $\left(\frac{3}{8}\right)^2$. We continue in this fashion, obtaining sets B_n consisting of 2^n disjoint closed intervals each of length $\left(\frac{3}{8}\right)^n$, and define our Cantor-like set $B = \bigcap_{n=0}^{\infty} B_n$.

(a) The set B is compact and perfect; the arguments used for the Cantor set work equally well for B. Each B_n is closed, being a finite union of closed intervals, and thus B is an intersection of closed sets and hence is itself closed. Certainly B is bounded and thus, by the Heine-Borel Theorem (Theorem 3.3.8), B is compact.

As in Exercise 3.4.3, given any $x \in B$ we can find a sequence of endpoints (x_n) such that $x_n \in B \setminus \{x\}$ and $|x - x_n| \leq \left(\frac{3}{8}\right)^n$ for each $n \in \mathbf{N}$. It follows from the Squeeze Theorem and Theorem 3.2.5 that x is a limit point of B and hence that B has no isolated points. Because B is also closed, we see that B is a perfect set.

(b) At the first stage, we remove an interval of length $\frac{1}{4}$. At the n^{th} stage $(n = 2, 3, 4, \ldots)$, we remove 2^{n-1} intervals each of length $\frac{1}{4}\left(\frac{3}{8}\right)^{n-1}$. Thus the length of B is

$$1 - \left(\frac{1}{4} + 2 \cdot \frac{1}{4} \cdot \frac{3}{8} + 2^2 \cdot \frac{1}{4} \cdot \left(\frac{3}{8}\right)^2 + \cdots\right)$$

$$= 1 - \frac{1}{4}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right) = 1 - \frac{\frac{1}{4}}{1 - \frac{3}{4}} = 0.$$

To calculate the dimension of B, we magnify the set by a factor of $\frac{8}{3}$, so that B_0 becomes the closed interval $\left[0, \frac{8}{3}\right]$. When we remove the open middle fourth of this interval, we are left with two intervals of length 1:

$$B_1 = [0, 1] \cup \left[\frac{5}{3}, \frac{8}{3}\right].$$

Continuing the construction, we will obtain two copies of B. The dimension x of B is then given by solving $2 = \left(\frac{8}{3}\right)^x$, which gives

$$x = \frac{\log(2)}{\log(8) - \log(3)} \approx 0.7067.$$

Exercise 3.4.5. Let A and B be nonempty subsets of $\mathbf{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Solution. Observe that V^c is a closed set which contains A (since $U \cap V = \emptyset$ implies that $A \cap V = \emptyset$). Since $\overline{A}$ is the smallest closed set containing A (Theorem 3.2.12), we must have $\overline{A} \subseteq V^c$, which gives

$$\overline{A} \subseteq V^c \quad \Rightarrow \quad \overline{A} \cap V = \emptyset \quad \Rightarrow \quad \overline{A} \cap B = \emptyset.$$

Similarly, $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

Exercise 3.4.6. Prove Theorem 3.4.6.

Solution. Suppose we have non-empty subsets $A, B \subseteq \mathbf{R}$ such that every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset. Since a limit point of A is the limit of a sequence of elements contained in A (Theorem 3.2.5) and an element of A is the limit of a constant sequence contained in A, and by assumption these limits do not belong to B, we see that $\overline{A} \cap B = \emptyset$. Similarly, $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

Conversely, suppose that A and B are separated. If $(x_n) \to x$ is a convergent sequence contained in A then $x \in \overline{A}$ and thus $x \notin B$ since $\overline{A} \cap B = \emptyset$. Similarly, the limit of any convergent sequence contained in B does not belong to A.

We have now shown that for non-empty subsets $A, B \subseteq \mathbf{R}$, A and B being separated is equivalent to the condition that every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset.

Proving Theorem 3.4.6 is equivalent to showing that a subset $E \subseteq \mathbf{R}$ is disconnected if and only if there exist non-empty subsets $A, B \subseteq E$ such that $E = A \cup B$ and every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset. By the previous discussion, such subsets are separated. So the theorem follows from the definition of disconnectedness.

Exercise 3.4.7. A set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A, y \in B$, and $E = A \cup B$.

(a) Show that $\mathbf{Q}$ is totally disconnected.

(b) Is the set of irrational numbers totally disconnected?

Solution.

(a) Suppose that $p < q$ are rational numbers. By the density of $\mathbf{I}$ in $\mathbf{R}$, there exists an irrational number y such that $p < y < q$. Define the sets

$$A = (-\infty, y) \cap \mathbf{Q} \quad \text{and} \quad B = (y, \infty) \cap \mathbf{Q}.$$

Notice that $p \in A, q \in B$, and $A \cup B = \mathbf{Q}$ since $y \notin \mathbf{Q}$. The density of $\mathbf{Q}$ in $\mathbf{R}$ implies that $\overline{A} = (-\infty, y]$ and $\overline{B} = [y, \infty)$. It follows that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and hence that A and B are separated. Thus $\mathbf{Q}$ is totally disconnected.

(b) $\mathbf{I}$ is also totally disconnected. To see this, reverse the roles of $\mathbf{Q}$ and $\mathbf{I}$ in the solution to part (a).

Exercise 3.4.8. Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7.

Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.

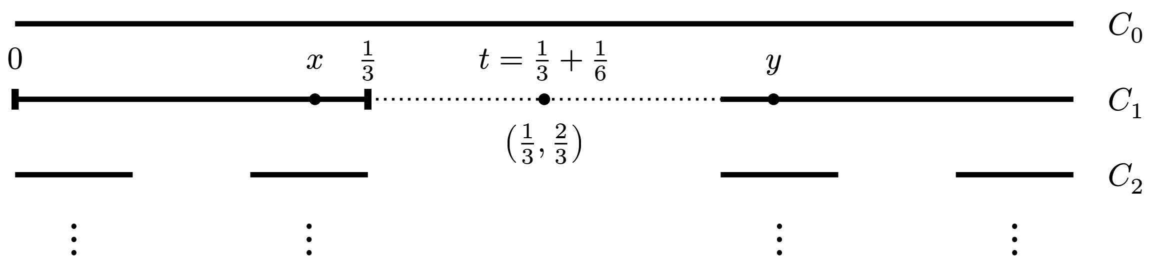
(a) Given $x, y \in C$, with $x < y$, set $\varepsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

(b) Show that C is totally disconnected.

Solution.

(a) If I is an interval of length δ, then any $a, b \in I$ must satisfy $|a - b| \leq \delta$. In the construction of C, each C_n consists of 2^n disjoint closed intervals each of length 3^{-n}. Thus we can find an N large enough so that C_N consists of closed intervals each of length $3^{-N} < \varepsilon = y - x$, i.e. whose length is smaller than the distance between x and y. It follows that x and y cannot possibly belong to the same interval of C_N.

(b) Let $[a, b]$ be the closed interval of C_N which contains x and note that the open interval $(b, b + 3^{-N})$ was either removed at the N^{th} stage of construction or is a subset of an open interval which was removed at some previous stage of construction. It follows that $t := b + \frac{1}{2} \cdot 3^{-N} \notin C$. Since $y \notin [a, b]$ and $y > x$, we must have $y > t$. Here is an example construction of t, with $N = 1$; notice that any $N \geq 1$ would also work.



Define $A = (-\infty, t) \cap C$ and $B = (t, \infty) \cap C$. Notice that $x \in A, y \in B$, and $A \cup B = C$ since $t \notin C$. Notice further that $\overline{A} \subseteq \overline{(-\infty, t)} = (-\infty, t]$ and similarly $\overline{B} \subseteq [t, \infty)$. It follows that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and hence that A and B are separated. Thus C is totally disconnected.

Exercise 3.4.9. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbf{N}$ set $\varepsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n)$ and let $F = O^c$.

(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.

(b) Does F contain any nonempty open intervals? Is F totally disconnected? (See Exercise 3.4.7 for the definition.)

(c) Is it possible to know whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

Solution.

(a) O is an open set since it is a union of open intervals, so $F = O^c$ must be closed. To see that F is non-empty, suppose otherwise, so that $O = \mathbf{R}$. It follows that the collection $\{V_{\varepsilon_n}(r_n) : n \in \mathbf{N}\}$ is an open cover of the compact set $[0, 10]$. Thus, by Theorem 3.3.8, there exist finitely many indices $n_1 < \cdots < n_\ell$ such that

$$[0, 10] \subseteq V_{\varepsilon_{n_1}}(r_{n_1}) \cup \cdots \cup V_{\varepsilon_{n_\ell}}(r_{n_\ell}).$$

However, the interval $[0, 10]$ has length 10, whereas the set $V_{\varepsilon_{n_1}}(r_{n_1}) \cup \cdots \cup V_{\varepsilon_{n_\ell}}(r_{n_\ell})$ has total length at most

$$\sum_{k=1}^{\ell} \frac{1}{2^{n_k - 1}} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2,$$

since $\left|V_{\varepsilon_{n_\ell}}(r_{n_\ell})\right| = 2\varepsilon_{n_k} = 2^{-n_k+1}$. So we have a set of length 10 contained inside a set of length 2, which is a contradiction; it follows that F is non-empty. Finally, since $\mathbf{Q} \subseteq O$, we see that $F = O^c$ can contain only irrational numbers.

(b) F cannot contain any non-empty open intervals, since this would imply that F contains a rational number (indeed, infinitely many rational numbers), but by part (a) F contains only irrational numbers.

To see that F is totally disconnected, let us prove the following lemma.

Lemma L.10. Suppose $G \subseteq \mathbf{R}$ is totally disconnected. If E is a non-empty subset of G, then E is also totally disconnected.

Proof. Let $x, y \in E$ be given. Since x and y belong to the totally disconnected set G, there exist separated sets A and B such that $x \in A, y \in B$, and $G = A \cup B$. Let $C = A \cap E$ and $D = B \cap E$ and note that $x \in C$ and $y \in D$. Furthermore, $C \subseteq A$ and $D \subseteq B$, so

$$\overline{C} \subseteq \overline{A} \quad \Rightarrow \quad \overline{C} \cap D \subseteq \overline{A} \cap D \subseteq \overline{A} \cap B = \emptyset.$$

Thus $\overline{C} \cap D = \emptyset$ and similarly $C \cap \overline{D} = \emptyset$; it follows that C and D are separated.

Finally,

$$E = E \cap G = E \cap (A \cup B) = (A \cap E) \cup (B \cap E) = C \cup D.$$

Thus E is totally disconnected. □

Since F is a subset of $\mathbf{I}$, which we showed was totally disconnected in Exercise 3.4.7, it follows from Lemma L.10 that F is totally disconnected.

(c) There are enumerations of $\mathbf{Q}$ which, when used in this construction, will result in an F which is not perfect, i.e. an F with at least one isolated point. Let y be an irrational number, say $y = \sqrt{2}$; we will construct an enumeration (r_n) of $\mathbf{Q}$, which gives an F with y as an isolated point, via the following four step process. (The idea for this construction comes from math.SE user Ingix.)

Step 1. First we will construct a sequence (p_n) of rational numbers with the following properties:

(1.1) $p_1 < p_2 < p_3 < \cdots < y$;

(1.2) $y \notin \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(p_n)$;

(1.3) $\left(y - \frac{1}{16}, y\right) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(p_n)$.

To define this sequence, for each $n \in \mathbf{N}$ let p_n be a rational number satisfying

$$y - \frac{1}{2^{4n}} - \frac{1}{2^{4n+4}} < p_n < y - \frac{1}{2^{4n}}, \quad \text{i.e.} \quad y - \varepsilon_{4n} - \varepsilon_{4n+4} < p_n < y - \varepsilon_{4n};$$

the existence of such a rational number is guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$. For any $n \in \mathbf{N}$ certainly $p_n < y$, and because $1 > \varepsilon_4 + \varepsilon_8$ we also have $\varepsilon_{4n} > \varepsilon_{4n+4} + \varepsilon_{4n+8}$, whence $p_n < p_{n+1}$. Thus (p_n) satisfies condition (1.1). Furthermore, for any $n \in \mathbf{N}$,

$$p_n + \varepsilon_{4n} < y \quad \Rightarrow \quad y \notin V_{\varepsilon_{4n}}(p_n).$$

Thus (p_n) satisfies condition (1.2). Notice that each p_{n+1} satisfies

$$p_n < p_{n+1} < p_n + \varepsilon_{4n} \quad \Rightarrow \quad p_{n+1} \in V_{\varepsilon_{4n}}(p_n),$$

i.e. the centre of $V_{\varepsilon_{4n+4}}(p_{n+1})$ is contained in $V_{\varepsilon_{4n}}(p_n)$. It follows that for any $N \in \mathbf{N}$ the union $\bigcup_{n=1}^{N} V_{\varepsilon_{4n}}(p_n)$ is an open interval:

$$\bigcup_{n=1}^{N} V_{\varepsilon_{4n}}(p_n) = \left(p_1 - \frac{1}{16}, B\right), \quad \text{where} \quad B = \max\{p_n + \varepsilon_{4n} : 1 \leq n \leq N\}.$$

(The exact value of B is not important, but note that it must be strictly less than y.) Observe that $y - \frac{1}{16} \in \bigcup_{n=1}^{N} V_{\varepsilon_{4n}}(p_n)$ for any $N \in \mathbf{N}$ since $y - \frac{1}{16} \in V_{\varepsilon_4}(p_1)$. Let $t \in \mathbf{R}$ be such that $y - \frac{1}{16} < t < y$. Because (p_n) converges to y, we can find an $N \in \mathbf{N}$ such that $t < p_N < y$. It follows that $y - \frac{1}{16}$ and p_N both belong to the open interval

$\bigcup_{n=1}^{N} V_{\varepsilon_{4n}}(p_n)$; since t lies between these two values, t must also belong to this open interval, i.e.

$$t \in \bigcup_{n=1}^{N} V_{\varepsilon_{4n}}(p_n) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(p_n).$$

Because this is true for any $t \in \left(y - \frac{1}{16}, y\right)$, we see that (p_n) satisfies condition (1.3).

Step 2. Now we will construct a sequence (q_n) of rational numbers with the following properties:

(2.1) $y < \cdots < q_3 < q_2 < q_1$;

(2.2) $y \notin \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n-2}}(q_n)$;

(2.3) $\left(y, y + \frac{1}{16}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n-2}}(q_n)$.

To define this sequence, for each $n \in \mathbf{N}$ let q_n be a rational number satisfying

$$y + \frac{1}{2^{4n-2}} < q_n < y + \frac{1}{2^{4n-2}} + \frac{1}{2^{4n+2}}, \quad \text{i.e.} \quad y + \varepsilon_{4n-2} < q_n < y + \varepsilon_{4n-2} + \varepsilon_{4n+2};$$

the existence of such a rational number is guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$. We can argue as in Step 1 to show that (q_n) satisfies conditions (2.1), (2.2), and (2.3).

Step 3. Since the sequences (p_n) and (q_n) constructed in Steps 1 and 2 are entirely contained inside the interval $[p_1, q_1]$, we still have infinitely many rational numbers left to enumerate. That is, letting

$$E = \mathbf{Q} \cap \left(\{p_1, p_2, \ldots\} \cup \{q_1, q_2, \ldots\}\right)^{\mathrm{c}},$$

we have that E is countably infinite. However, enumerating E carelessly might exclude y from F in Step 4, since there are rational numbers in E arbitrarily close to y; placing one of these rational numbers "too early" in the final enumeration will include y in the ε_n-neighbourhood of that rational number. To surmount this problem, we will construct an enumeration (a_n) of E with the following property:

(3.1) $y \notin V_{\varepsilon_{2n-1}}(a_n)$ for all $n \in \mathbf{N}$.

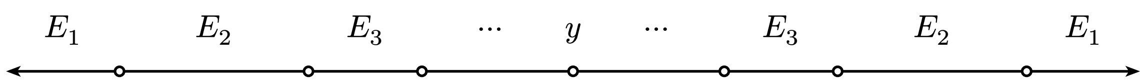
We will first partition E as follows. For each $n \in \mathbf{N}$, let

$$A_n = \begin{cases} \{x \in \mathbf{R} : \varepsilon_1 < |x - y|\} & \text{if } n = 1, \\ \{x \in \mathbf{R} : \varepsilon_{2n-1} < |x - y|\} < \varepsilon_{2n-3} & \text{if } n \geq 2. \end{cases}$$

Equivalently,

$$A_n = \begin{cases} (-\infty, y - \varepsilon_1) \cup (y + \varepsilon_1, \infty) & \text{if } n = 1, \\ (y - \varepsilon_{2n-3}, y - \varepsilon_{2n-1}) \cup (y + \varepsilon_{2n-1}, y + \varepsilon_{2n-3}) & \text{if } n \geq 2. \end{cases}$$

Now let $E_n = E \cap A_n$ for each $n \in \mathbf{N}$.



We have $\bigcup_{n=1}^{\infty} E_n = E$ since the only real numbers not contained in $\bigcup_{n=1}^{\infty} A_n$ are y and those of the form $y \pm \varepsilon_{2n-1}$ for some $n \in \mathbf{N}$, none of which is rational, and the collection $\{E_n : n \in \mathbf{N}\}$ is evidently pairwise disjoint; it follows that this collection is a partition of E.

Because $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = y$ and $y \notin \overline{A_n}$ for any $n \in \mathbf{N}$, there can be only finitely many terms of the sequences (p_n) and (q_n) contained in each A_n. Thus each E_n is countably infinite. We can then enumerate each E_n:

$$E_n = \{e_{1,n}, e_{2,n}, e_{3,n}, \dots\}.$$

These enumerations can be combined to form an enumeration (a_n) of E using the same method used in the proof that a countable union of countable sets is countable (see Exercise 1.5.3 (c)). To be precise, consider the following "infinite arrays".

E_1	E_2	E_3	E_4	E_5	$\cdots$
$e_{1,1}$	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$	$e_{1,5}$	$\cdots$
$e_{2,1}$	$e_{2,2}$	$e_{2,3}$	$e_{2,4}$	$\ddots$	
$e_{3,1}$	$e_{3,2}$	$e_{3,3}$	$\ddots$		
$e_{4,1}$	$e_{4,2}$	$\ddots$			
$e_{5,1}$	$\ddots$				
$\vdots$					

1	2	3	4	5	$\cdots$
a_1	a_3	a_6	a_{10}	a_{15}	$\cdots$
a_2	a_5	a_9	a_{14}	$\ddots$	
a_4	a_8	a_{13}	$\ddots$		
a_7	a_{12}	$\ddots$			
a_{11}	$\ddots$				
$\vdots$					

The enumeration of E_n is the n^{th} column of the left-hand array. The enumeration of E is obtained by letting a_N in the right-hand array be the element $e_{m,n}$ in the corresponding position of the left-hand array, so that

$$a_1 = e_{1,1}, \quad a_2 = e_{2,1}, \quad a_3 = e_{1,2}, \quad a_4 = e_{3,1}, \quad \dots$$

This mapping is bijective because the collection $\{E_n : n \in \mathbf{N}\}$ is a partition of E. Now we need to show that (a_n) satisfies condition (3.1). Let $n \in \mathbf{N}$ be given. The element a_n belongs to some column of the right-hand array above, say the N^{th} column. From the definition of our enumeration (a_n), we have $a_n = e_{m,N}$ for some $m \in \mathbf{N}$. It follows that $a_n \in E_N$ and hence that $|a_n - y| > \varepsilon_{2N-1}$, which gives $y \notin V_{\varepsilon_{2N-1}}(a_n)$. If we examine the right-hand array, we see that the element at the top of the N^{th} column is $a_{N(N+1)/2}$ (the N^{th} triangular number), and furthermore that $n \geq N(N+1)/2$. Thus

$$2n - 1 \geq 2N - 1 \quad \Rightarrow \quad \varepsilon_{2n-1} \leq \varepsilon_{2N-1} \quad \Rightarrow \quad V_{\varepsilon_{2n-1}}(a_n) \subseteq V_{\varepsilon_{2N-1}}(a_n).$$

Combining this with $y \notin V_{\varepsilon_{2N-1}}(a_n)$, we see that $y \notin V_{\varepsilon_{2n-1}}(a_n)$. Thus (a_n) satisfies condition (3.3).

Step 4. We can now form our final enumeration (r_n) of $\mathbf{Q}$, by letting

$$r_{2n-1} = a_n, \quad r_{4n-2} = q_n, \quad \text{and} \quad r_{4n} = p_n,$$

so that $(r_n) = (a_1, q_1, a_2, p_1, a_3, q_2, a_4, p_2, ...)$. Let $O = \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n)$ and $F = O^{\mathrm{c}}$. By condition (1.2), we have

$$\left(y - \frac{1}{16}, y\right) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(p_n) = \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(r_{4n}) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n) = O,$$

and by condition (2.2) we have

$$\left(y, y + \frac{1}{16}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n-2}}(q_n) = \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n-2}}(r_{4n-2}) \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n) = O.$$

Thus $\left(y - \frac{1}{16}, y\right) \cup \left(y, y + \frac{1}{16}\right) \subseteq O$. Furthermore, since

$$\begin{aligned} O &= \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n) \\ &= \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(r_{4n}) \cup \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n-2}}(r_{4n-2}) \cup \bigcup_{n=1}^{\infty} V_{\varepsilon_{2n-1}}(r_{2n-1}) \\ &= \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n}}(p_n) \cup \bigcup_{n=1}^{\infty} V_{\varepsilon_{4n-2}}(q_n) \cup \bigcup_{n=1}^{\infty} V_{\varepsilon_{2n-1}}(a_n), \end{aligned}$$

conditions (1.3), (2.3), and (3.1) imply that $y \notin O$. It follows that

$$\left(y - \frac{1}{16}, y + \frac{1}{16}\right) \cap F = \{y\},$$

so that y is an isolated point of F. We may conclude that F is not a perfect set.

Regarding the second half of the question, it is possible to modify the construction to produce a non-empty perfect set consisting of only irrational numbers. To do this, we start with any enumeration (r_n) of $\mathbf{Q}$ and inductively define a sequence of non-negative real numbers (ε_n) in such a way that if let

$$O = \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n) \quad \text{and} \quad F = O^{\mathrm{c}},$$

then F will be a non-empty perfect set of irrational numbers. Intuitively, we will inductively construct O as a union of disjoint open intervals, with no pair of these intervals sharing an endpoint. (In what follows, we adopt the convention that $V_\varepsilon(x) = \emptyset$ if $\varepsilon = 0$.)

Suppose that after N steps we have chosen $\varepsilon_1, ..., \varepsilon_N$ such that:

(IH1) $\{r_1, ..., r_N\} \subseteq \bigcup_{n=1}^{N} V_{\varepsilon_n}(r_n)$;

(IH2) for all $1 \leq n \leq N$, either $\varepsilon_n = 0$ or ε_n is irrational and satisfies $0 < \varepsilon_n \leq 2^{-n}\sqrt{2}$;

(IH3) $\overline{V_{\varepsilon_m}(r_m)} \cap \overline{V_{\varepsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N$.

Let $U = \bigcup_{n=1}^{N} V_{\varepsilon_n}(r_n)$. There are two cases.

Case 1. This is the easier case. If $r_{N+1} \in U$ then let $\varepsilon_{N+1} = 0$, so that $V_{\varepsilon_{N+1}}(r_{N+1}) = \emptyset$. Combining this with (IH1) gives us

$$\{r_1, ..., r_N, r_{N+1}\} \subseteq U = \bigcup_{n=1}^{N} V_{\varepsilon_n}(r_n) = \bigcup_{n=1}^{N+1} V_{\varepsilon_n}(r_n).$$

(IH2) together with $\varepsilon_{N+1} = 0$ shows that for all $1 \leq n \leq N+1$, either $\varepsilon_n = 0$ or ε_n is irrational and satisfies $0 < \varepsilon_n \leq 2^{-n}\sqrt{2}$.

Similarly, combining (IH3) with $V_{\varepsilon_{N+1}}(r_{N+1}) = \emptyset$, we have $\overline{V_{\varepsilon_m}(r_m)} \cap \overline{V_{\varepsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N+1$.

Case 2. This is the harder case. If $r_{N+1} \notin U$ then let $\varepsilon_{n_1}, ..., \varepsilon_{n_J}$ be those ε's from $\varepsilon_1, ..., \varepsilon_N$ which are non-zero; there must be at least one such ε_{n_j} by (IH1) and each ε_{n_j} must be positive and irrational by (IH2). Observe that

$$U = \bigcup_{n=1}^{N} V_{\varepsilon_n}(r_n) = \bigcup_{j=1}^{J} V_{\varepsilon_{n_j}}(r_{n_j}),$$

where each $V_{\varepsilon_{n_j}}(r_{n_j})$ is a proper open interval. For each $1 \leq j \leq J$, note that since $r_{N+1} \notin U$, we must have $r_{N+1} \notin V_{\varepsilon_{n_j}}(r_{n_j})$. Both of the endpoints of $V_{\varepsilon_{n_j}}(r_{n_j})$ are the sum of a rational number and an irrational number and hence are irrational; since r_{N+1} is rational, we see that $r_{N+1} \notin [r_{n_j} - \varepsilon_{n_j}, r_{n_j} + \varepsilon_{n_j}]$. Given this, if we let d be the minimum of the distances from r_{N+1} to the endpoints of each $V_{\varepsilon_{n_j}}$, i.e.

$$d = \min\left\{|r_{n_j} - \varepsilon_{n_j} - r_{N+1}|, |r_{n_j} + \varepsilon_{n_j} - r_{N+1}| : 1 \leq j \leq J\right\},$$

then d must be positive. Furthermore, d must be irrational since it is the sum of a rational number and an irrational number, and for each $1 \leq j \leq J$ we have

$$\left[r_{N+1} - \tfrac{d}{2}, r_{N+1} + \tfrac{d}{2}\right] \cap [r_{n_j} - \varepsilon_{n_j}, r_{n_j} + \varepsilon_{n_j}] = \emptyset. \qquad (*)$$

Let $\varepsilon_{N+1} = \min\left\{2^{-(N+1)}\sqrt{2}, \frac{d}{2}\right\}$ and note that ε_{N+1} is positive, so that $r_{N+1} \in V_{\varepsilon_{N+1}}(r_{N+1})$. Combining this with (IH1) gives us

$$\{r_1, ..., r_N, r_{N+1}\} \subseteq \bigcup_{n=1}^{N+1} V_{\varepsilon_n}(r_n).$$

As noted before, d is positive and irrational, so ε_{N+1} is positive, irrational, and satisfies $\varepsilon_{N+1} \leq 2^{-(N+1)}\sqrt{2}$; combining this with (IH1) shows that for all $1 \leq n \leq N+1$, either $\varepsilon_n = 0$ or ε_n is irrational and satisfies $0 < \varepsilon_n \leq 2^{-n}\sqrt{2}$.

Let $1 \leq n \leq N$ be given. If $\varepsilon_n = 0$ then the identity $\overline{V_{\varepsilon_n}(r_n)} \cap \overline{V_{\varepsilon_{N+1}}(r_{N+1})} = \emptyset$ is clear, since $V_{\varepsilon_n}(r_n) = \emptyset$. If $\varepsilon_n \neq 0$ then $n = n_j$ for some $1 \leq j \leq J$. In this case, we have

$$\overline{V_{\varepsilon_n}(r_n)} = \overline{V_{\varepsilon_{n_j}}(r_{n_j})} = [r_{n_j} - \varepsilon_{n_j}, r_{n_j} + \varepsilon_{n_j}] \quad \text{and}$$

$$\overline{V_{\varepsilon_{N+1}}(r_{N+1})} = [r_{N+1} - \varepsilon_{N+1}, r_{N+1} + \varepsilon_{N+1}] \subseteq \left[r_{N+1} - \tfrac{d}{2}, r_{N+1} + \tfrac{d}{2}\right].$$

It then follows from equation $(*)$ that $\overline{V_{\varepsilon_n}(r_n)} \cap \overline{V_{\varepsilon_{N+1}}(r_{N+1})} = \emptyset$. Combining this with (IH3), we see that $\overline{V_{\varepsilon_m}(r_m)} \cap \overline{V_{\varepsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \le m < n \le N+1$.

This completes the inductive step; for the base case, simply let $\varepsilon_1 = \frac{\sqrt{2}}{2}$. By induction we obtain a sequence (ε_n) which satisfies (IH1), (IH2), and (IH3) for all $N \in \mathbf{N}$. In other words, the sequence (ε_n) has the following properties:

(A1) $\mathbf{Q} \subseteq \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n)$;

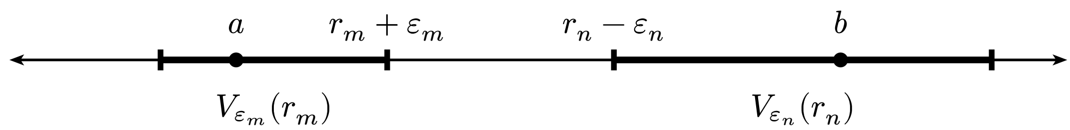
(A2) for all $n \in \mathbf{N}$, either $\varepsilon_n = 0$ or ε_n is irrational and satisfies $0 < \varepsilon_n \le 2^{-n}\sqrt{2}$;

(A3) $\overline{V_{\varepsilon_m}(r_m)} \cap \overline{V_{\varepsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $m < n$.

Let $O = \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n)$ and $F = O^{\mathrm{c}}$. As in part (a), F is closed and, by (A1), consists solely of irrational numbers. By (A2) we have $\varepsilon_n \le 2^{-n}\sqrt{2}$ for each $n \in \mathbf{N}$; an argument similar to the one given in part (a) shows that O cannot be the entire real line and thus F is non-empty.

To see that F is perfect, suppose by way of contradiction that $x \in F$ is isolated, i.e. there exists a $\delta > 0$ such that $(x-\delta, x+\delta) \cap F = \{x\}$. This implies that the intervals $(x-\delta, x)$ and $(x, x+\delta)$ are contained in O. We claim that if an open interval (c, d) is to be contained in O, then it must be entirely contained inside a single $V_{\varepsilon_n}(r_n)$. To see this, suppose by way of contradiction that $a, b \in (c, d)$ are such that $a < b$, $a \in V_{\varepsilon_m}(r_m)$, and $b \in V_{\varepsilon_n}(r_n)$, with $m \ne n$. By (A3), it must then be the case that

$$a < r_m + \varepsilon_m < r_n - \varepsilon_n < b.$$



So $r_m + \varepsilon_m \in (a, b) \subseteq (c, d) \subseteq O$; it follows that there exists some $k \in \mathbf{N}$ such that $r_m + \varepsilon_m$ belongs to $V_{\varepsilon_k}(r_k)$. If $k = m$ this says that an open interval contains one of its endpoints, and if $k \ne m$ then this violates (A3). In either case, we have a contradiction.

Thus any open interval (c, d) contained in O must be entirely contained inside a single $V_{\varepsilon_n}(r_n)$. Since $(x-\delta, x)$ and $(x, x+\delta)$ are disjoint, there exist positive integers $m \ne n$ such that

$$(x-\delta, x) \subseteq V_{\varepsilon_m}(r_m) \quad \text{and} \quad (x, x+\delta) \subseteq V_{\varepsilon_n}(r_n).$$

This implies that

$$[x-\delta, x] \subseteq \overline{V_{\varepsilon_m}(r_m)} \quad \text{and} \quad [x, x+\delta] \subseteq \overline{V_{\varepsilon_n}(r_n)},$$

which gives us $x \in \overline{V_{\varepsilon_m}(r_m)} \cap \overline{V_{\varepsilon_n}(r_n)}$, contradicting (A3). We may conclude that F contains no isolated points, i.e. F is a perfect set.

3.5. Baire's Theorem

Exercise 3.5.1. Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

Solution. This is immediate from De Morgan's Laws (see Exercise 3.2.9).

Exercise 3.5.2. Replace each ______ with the word *finite* or *countable* depending on which is more appropriate.

(a) The ______ union of F_σ sets is an F_σ set.

(b) The ______ of F_σ sets is an F_σ set.

(c) The ______ union of G_δ sets is a G_δ set.

(d) The ______ intersection of G_δ sets is a G_δ set.

Solution.

(a) The countable union of F_σ sets is an F_σ set. Suppose we have a countable collection $\{A_m : m \in \mathbf{N}\}$ of F_σ sets, i.e. for each $m \in \mathbf{N}$ there is a countable collection $\left\{B_{m,n} : n \in \mathbf{N}\right\}$ of closed sets such that $A_m = \bigcup_{n=1}^{\infty} B_{m,n}$. Notice that

$$\bigcup_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} B_{m,n}.$$

Because $\mathbf{N}^2$ is countable (Lemma L.5), the expression above shows that $\bigcup_{m=1}^{\infty} A_m$ is a countable union of closed sets; it follows that $\bigcup_{m=1}^{\infty} A_m$ is an F_σ set.

(b) The finite intersection of F_σ sets is an F_σ set. To see this, it will suffice to show that if A and B are F_σ sets, then $A \cap B$ is an F_σ set; the general case will then follow from a straightforward induction argument. Suppose therefore that $A = \bigcup_{m=1}^{\infty} A_m$ and $B = \bigcup_{n=1}^{\infty} B_n$, where $\{A_m : m \in \mathbf{N}\}$ and $\{B_n : n \in \mathbf{N}\}$ are countable collections of closed sets, and observe that

$$A \cap B = \left(\bigcup_{m=1}^{\infty} A_m\right) \cap \left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (A_m \cap B_n).$$

Since each $A_m \cap B_n$ is closed (being an intersection of closeed sets) and $\mathbf{N}^2$ is countable (Lemma L.5), we have expressed $A \cap B$ as a countable union of closed sets; it follows that $A \cap B$ is an F_σ set.

The countable intersection of F_σ sets need not be an F_σ set. For a counterexample, let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbf{Q}$ and for positive integers m, n, let

$$B_{m,n} = \left(-\infty, r_m - \frac{1}{n}\right] \cup \left[r_m + \frac{1}{n}, \infty\right).$$

Each $B_{m,n}$ is a closed set, so if we let $A_m = \bigcup_{n=1}^{\infty} B_{m,n}$ for each $m \in \mathbf{N}$ then each A_m is an F_σ set. We claim that $\bigcap_{m=1}^{\infty} A_m = \mathbf{I}$, the set of irrational numbers. To see this, we will show that $\left(\bigcap_{m=1}^{\infty} A_m\right)^{\mathrm{c}} = \mathbf{Q}$. By De Morgan's Laws (Exercise 3.2.9), we have

$$\left(\bigcap_{m=1}^{\infty} A_m\right)^{\mathrm{c}} = \bigcup_{m=1}^{\infty} A_m^{\mathrm{c}} = \bigcup_{m=1}^{\infty} \left(\bigcup_{n=1}^{\infty} B_{m,n}\right)^{\mathrm{c}}$$

$$= \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_{m,n}^{\mathrm{c}} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left(r_m - \frac{1}{n}, r_m + \frac{1}{n}\right) = \bigcup_{m=1}^{\infty} \{r_m\} = \mathbf{Q}.$$

Thus $\bigcap_{m=1}^{\infty} A_m = \mathbf{I}$. As we will show in Exercise 3.5.6, $\mathbf{I}$ is not an F_σ set.

(c) The finite union of G_δ sets is a G_δ set, but the countable union of G_δ sets need not be a G_δ set; these statements follow from part (b) of this exercise, Exercise 3.5.1, and De Morgan's Laws (Exercise 3.2.9).

(d) The countable intersection of G_δ sets is a G_δ set. This follows from part (a) of this exercise, Exercise 3.5.1, and De Morgan's Laws (Exercise 3.2.9).

Exercise 3.5.3. (This exercise has already appeared as Exercise 3.2.15.)

(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set.

Solution. See Exercise 3.2.15.

Exercise 3.5.4. Starting with $n = 1$, inductively construct a nested sequence of *closed* intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue of the endpoints of each I_n. Show how this leads to a proof of the theorem.

Solution. Since G_1 is dense it must be non-empty, i.e. there exists some $x_1 \in G_1$, and then since G_1 is open there exists an $\varepsilon_1 > 0$ such that $V_{\varepsilon_1}(x_1) \subseteq G_1$. Let

$$a_1 = x_1 - \frac{\varepsilon_1}{2}, \quad b_1 = x_1 + \frac{\varepsilon_1}{2}, \quad \text{and} \quad I_1 = [a_1, b_1],$$

and note that $I_1 \subseteq V_{\varepsilon_1}(x_1) \subseteq G_1$. This handles the base case.

Suppose that after n steps we have chosen nested, closed intervals

$$I_1 = [a_1, b_1] \supseteq \cdots \supseteq I_n = [a_n, b_n]$$

such that $I_1 \subseteq G_1, \dots, I_n \subseteq G_n$ and $a_1 < b_1, \dots, a_n < b_n$. Because G_{n+1} is dense there exists some $x_{n+1} \in G_{n+1}$ such that $a_n < x_{n+1} < b_n$, and since G_{n+1} is open there exists some $\varepsilon_{n+1} > 0$ such that $V_{\varepsilon_{n+1}}(x_{n+1}) \subseteq G_{n+1}$. Let $\delta = \min\{2^{-1}\varepsilon_{n+1}, x_{n+1} - a_n, b_n - x_{n+1}\}$ and define

$$a_{n+1} = x_{n+1} - \delta, \quad b_{n+1} = x_{n+1} + \delta, \quad \text{and} \quad I_{n+1} = [a_{n+1}, b_{n+1}].$$

Note that $a_{n+1} < b_{n+1}$, and since $\delta \le x_{n+1} - a_n$ and $\delta \le b_n - x_{n+1}$ we have $I_{n+1} \subseteq I_n$. Moreover, because $\delta \le 2^{-1}\varepsilon_{n+1}$, we also have $I_{n+1} \subseteq V_{\varepsilon_{n+1}}(x_{n+1}) \subseteq G_{n+1}$. This completes the induction step.

Via induction we obtain a nested sequence of closed intervals $(I_n)_{n=1}^{\infty}$ such that $I_n \subseteq G_n$ for each $n \in \mathbf{N}$. We may now appeal to the Nested Interval Property (Theorem 1.4.1) to obtain some $x \in \bigcap_{n=1}^{\infty} I_n$, which must also belong to $\bigcap_{n=1}^{\infty} G_n$.

Exercise 3.5.5. Show that it is impossible to write

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbf{N}$, F_n is a closed set containing no nonempty open intervals.

Solution. Suppose that $\{F_n : n \in \mathbf{N}\}$ is a collection of closed sets, each of which contains no non-empty open intervals. Let $n \in \mathbf{N}$ be given and let $x < z$ be arbitrary real numbers. By assumption $(x, z) \nsubseteq F_n$, so there must exist some $y \in (x, z) \cap F_n^{c}$; it follows that F_n^{c} is dense. Thus $\{F_n^{c} : n \in \mathbf{N}\}$ is a collection of open, dense sets. Theorem 3.5.2 and De Morgan's Laws (Exercise 3.2.9) now imply that

$$\bigcap_{n=1}^{\infty} F_n^{c} \neq \emptyset \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} F_n \neq \mathbf{R}.$$

Exercise 3.5.6. Show how the previous exercise implies that the set $\mathbf{I}$ of irrationals cannot be an F_σ set, and $\mathbf{Q}$ cannot be a G_δ set.

Solution. We will argue by contradiction. Suppose that $\mathbf{I}$ is an F_σ set, so that $\mathbf{I} = \bigcup_{m=1}^{\infty} F_m$, where each F_m is closed. Note that for any $m \in \mathbf{N}$, it must be the case that F_m contains no non-empty open interval; otherwise, F_m would contain infinitely many rational numbers. Let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbf{Q}$, so that $\mathbf{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$, and note that each singleton $\{r_n\}$ is closed and contains no non-empty open interval. Note further that

$$\mathbf{R} = \mathbf{I} \cup \mathbf{Q} = \left(\bigcup_{m=1}^{\infty} F_m \right) \cup \left(\bigcup_{n=1}^{\infty} \{r_n\} \right) = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (F_m \cup \{r_n\}).$$

For any $m, n \in \mathbf{N}$ the union $F_m \cup \{r_n\}$ is closed and contains no non-empty open intervals. However, since $\mathbf{N}^2$ is countable (Lemma L.5), this expression for $\mathbf{R}$ contradicts Exercise 3.5.5. Thus it must be the case that $\mathbf{I}$ is not an F_σ set, which by Exercise 3.5.1 implies that $\mathbf{Q}$ cannot be a G_δ set.

Exercise 3.5.7. Using Exercise 3.5.6 and versions of the statements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

Solution. Define $E = (\mathbf{I} \cap (-\infty, 0]) \cup (\mathbf{Q} \cap [0, \infty))$; we claim that E is neither an F_σ nor a G_δ set. Seeking a contradiction, suppose that E is an F_σ set. Notice that $(-\infty, 0)$ is an F_σ set:

$$(-\infty, 0) = \bigcup_{n=1}^{\infty} \Big(-\infty, -\frac{1}{n}\Big].$$

It follows from Exercise 3.5.2 (b) that

$$E \cap (-\infty, 0) = \mathbf{I} \cap (-\infty, 0)$$

is an F_σ set, i.e. there is a countable collection $\{F_m : m \in \mathbf{N}\}$ of closed sets such that

$$\mathbf{I} \cap (-\infty, 0) = \bigcup_{m=1}^{\infty} F_m.$$

For $m \in \mathbf{N}$, let $-F_m = \{-x : x \in F_m\}$. Since $(x_n) \to x$ implies $(-x_n) \to -x$, each $-F_m$ is closed. Furthermore,

$$\mathbf{I} \cap (0, \infty) = \bigcup_{m=1}^{\infty} -F_m.$$

It follows that $\mathbf{I} \cap (0, \infty)$ is an F_σ set. However, Exercise 3.5.2 (a) now implies that

$$\mathbf{I} = (\mathbf{I} \cap (-\infty, 0)) \cup (\mathbf{I} \cap (0, \infty))$$

is an F_σ set, contradicting Exercise 3.5.6. Thus E cannot be an F_σ set.

Seeking another contradiction, suppose that E is a G_δ set. Notice that $[0, \infty)$ is a G_δ set:

$$[0, \infty) = \bigcap_{n=1}^{\infty} \Big(-\frac{1}{n}, \infty\Big).$$

It follows from Exercise 3.5.2 (b) that

$$E \cap [0, \infty) = \mathbf{Q} \cap [0, \infty)$$

is a G_δ set, i.e. there is a countable collection $\{O_m : m \in \mathbf{N}\}$ of open sets such that

$$\mathbf{Q} \cap [0, \infty) = \bigcap_{m=1}^{\infty} O_m.$$

For $m \in \mathbf{N}$, let $-O_m = \{-x : x \in O_m\}$. Because $V_\varepsilon(x) \subseteq O_m$ implies $V_\varepsilon(-x) \subseteq -O_m$ for any $\varepsilon > 0$, each $-O_m$ is open. Furthermore,

$$\mathbf{Q} \cap (0, \infty] = \bigcap_{m=1}^{\infty} -O_m.$$

It follows that $\mathbf{Q} \cap (0, \infty]$ is a G_δ set. However, Exercise 3.5.2 (c) now implies that

$$\mathbf{Q} = (\mathbf{Q} \cap (-\infty, 0]) \cup (\mathbf{Q} \cap [0, \infty))$$

is a G_δ set, contradicting Exercise 3.5.6. Thus E cannot be a G_δ set.

Exercise 3.5.8. Show that a set E is nowhere-dense in $\mathbf{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbf{R}$.

Solution. We will show that $A \subseteq \mathbf{R}$ contains no non-empty open intervals if and only if A^c is dense in $\mathbf{R}$; the desired result can then be obtained by taking $A = \overline{E}$. By A containing no non-empty open intervals, we mean that for all $x, y \in \mathbf{R}$ such that $x < y$, we have $(x, y) \not\subseteq A$. This is equivalent to saying that for all $x, y \in \mathbf{R}$ such that $x < y$, there exists some $t \in \mathbf{R}$ such that $x < t < y$ and $t \notin A$. In other words, A^c is dense in $\mathbf{R}$.

Exercise 3.5.9. Decide whether the following sets are dense in $\mathbf{R}$, nowhere-dense in $\mathbf{R}$, or somewhere in between.

(a) $A = \mathbf{Q} \cap [0, 5]$.

(b) $B = \{1/n : n \in \mathbf{N}\}$.

(c) the set of irrationals.

(d) the Cantor set.

Solution.

(a) We have $\overline{A} = [0, 5]$, which is not the entire real line and also contains non-empty open intervals. Thus A is neither dense nor nowhere-dense.

(b) We have $\overline{B} = \{0\} \cup B \neq \mathbf{R}$, so that B is not dense. Note that if $\overline{B}$ contained a non-empty open interval then $\overline{B}$ would contain at least one irrational number, but $\overline{B} \subseteq \mathbf{Q}$. Thus $\overline{B}$ contains no non-empty open intervals and it follows that B is nowhere-dense.

(c) $\mathbf{I}$ is dense in $\mathbf{R}$ (see Exercise 1.4.5) and hence cannot be nowhere-dense (a dense subset $E \subseteq \mathbf{R}$ certainly cannot be nowhere-dense; $\overline{E} \subseteq \mathbf{R}$ contains every non-empty open interval).

(d) The Cantor set is closed, so $\overline{C} = C \neq \mathbf{R}$; it follows that C is not dense in $\mathbf{R}$. Furthermore, C does not contain any non-empty open intervals; given any $x < y \in C$, it is always possible to find some $t \notin C$ such that $x < t < y$ (see Exercise 3.4.8). Thus C is nowhere-dense in $\mathbf{R}$.

Exercise 3.5.10. Finish the proof by finding a contradiction to the results in this section.

Solution. Since $E_n \subseteq \overline{E_n}$ for each $n \in \mathbf{N}$, we have $\mathbf{R} = \bigcup_{n=1}^\infty \overline{E_n}$. However, each $\overline{E_n}$ is closed and by assumption contains no non-empty open intervals, so this contradicts Exercise 3.5.5.

Chapter 4. Functional Limits and Continuity

4.2. Functional Limits

Exercise 4.2.1.

(a) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

Solution.

(a) Suppose (x_n) is a sequence contained in A, satisfying $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. The sequential criterion (Theorem 4.2.3) implies that

$$\lim_{n\to\infty} f(x_n) = L \quad \text{and} \quad \lim_{n\to\infty} g(x_n) = M,$$

and thus the Algebraic Limit Theorem (for sequences, Theorem 2.3.3) gives

$$\lim_{n\to\infty} [f(x_n) + g(x_n)] = L + M.$$

The sequential criterion allows us to conclude that $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(b) Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist positive real numbers δ_1 and δ_2 such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \quad \Rightarrow \quad |f(x) - L| < \frac{\varepsilon}{2},$$

$$0 < |x - c| < \delta_2 \text{ and } x \in A \quad \Rightarrow \quad |g(x) - M| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $x \in A$ is such that $0 < |x - c| < \delta$. Observe that

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(c) Suppose (x_n) is a sequence contained in A, satisfying $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. The sequential criterion (Theorem 4.2.3) implies that

$$\lim_{n\to\infty} f(x_n) = L \quad \text{and} \quad \lim_{n\to\infty} g(x_n) = M,$$

and thus the Algebraic Limit Theorem (Theorem 2.3.3) gives

$$\lim_{n\to\infty}[f(x_n)g(x_n)] = LM.$$

The sequential criterion allows us to conclude that $\lim_{x\to c}[f(x)g(x)] = LM$.

Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist positive real numbers δ_1, δ_2, and δ_3 such that

$$0 < |x-c| < \delta_1 \text{ and } x \in A \quad \Rightarrow \quad |f(x) - L| < \frac{\varepsilon}{2(|M|+1)},$$

$$0 < |x-c| < \delta_2 \text{ and } x \in A \quad \Rightarrow \quad |g(x) - M| < \frac{\varepsilon}{2(|L|+1)},$$

$$0 < |x-c| < \delta_3 \text{ and } x \in A \quad \Rightarrow \quad |g(x) - M| < 1 \quad \Rightarrow \quad |g(x)| < |M| + 1.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, suppose that $x \in A$ is such that $0 < |x - c| < \delta$, and observe that

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |g(x)[f(x) - L] + L[g(x) - M]| \\
&\le |g(x)||f(x) - L| + |L||g(x) - M| \\
&< (|M|+1)\frac{\varepsilon}{2(|M|+1)} + |L|\frac{\varepsilon}{2(|L|+1)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $\lim_{x\to c}[f(x)g(x)] = LM$.

Exercise 4.2.2. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ε challenge.

(a) $\lim_{x\to 3}(5x - 6) = 9$, where $\varepsilon = 1$.

(b) $\lim_{x\to 4}\sqrt{x} = 2$, where $\varepsilon = 1$.

(c) $\lim_{x\to \pi}[[x]] = 3$, where $\varepsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or equal to x.)

(d) $\lim_{x\to \pi}[[x]] = 3$, where $\varepsilon = .01$.

Solution.

(a) Observe that

$$|5x - 6 - 9| = 5|x - 3| < 1 \quad \Leftrightarrow \quad |x - 3| < \frac{1}{5}.$$

Thus $\delta = \frac{1}{5}$ is the largest possible δ we can take.

(b) It is straightforward to verify that $x \in (1,7) = V_3(4)$ gives us $\sqrt{x} \in (1,3) = V_1(2)$, so that $\delta = 3$ is a valid response to $\varepsilon = 1$. No larger value of δ will work, since this would give us an $x \in [0,1)$, which implies $\sqrt{x} \in [0,1) \nsubseteq (1,3)$.

(c) Since $[[x]]$ is always an integer, we have $|[[x]] - 3| < 1$ if and only if $[[x]] = 3$, which is the case if and only if $3 \leq x < 4$. Thus we should choose the largest possible δ such that $V_\delta(\pi) \subseteq [3,4)$, which is

$$\delta = \min\{\pi - 3, 4 - \pi\} = \pi - 3.$$

(d) As in part (c), we have $|[[x]] - 3| < 0.01$ if nad only if $[[x]] = 3$, so the largest possible choice is $\delta = \pi - 3$.

Exercise 4.2.3. Review the definition of Thomae's function $t(x)$ from Section 4.1.

(a) Construct three different sequences $(x_n), (y_n)$, and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

(b) Now, compute $\lim t(x_n), \lim t(y_n)$, and $\lim t(z_n)$.

(c) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1B to verify the claim. (Given $\varepsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \varepsilon\}$. Argue that all the points in this set are isolated.)

Solution.

(a) We can take

$$x_n = 1 + \frac{1}{n}, \quad y_n = 1 - \frac{1}{n}, \quad \text{and} \quad z_n = 1 + \frac{\sqrt{2}}{n}.$$

(b) Since $x_n = \frac{n+1}{n}$, we have $t(x_n) = \frac{1}{n}$ and thus $\lim t(x_n) = 0$. Similarly, $y_n = \frac{n-1}{n}$, so $t(y_1) = t(0) = 1$ and $t(y_n) = \frac{1}{n}$ for $n \geq 2$. Thus $\lim t(y_n) = 0$ also. Finally, since $z_n \in \mathbf{I}$ for each $n \in \mathbf{N}$, we have $\lim t(z_n) = 0$.

(c) We conjecture that $\lim_{x\to 1} t(x) = 0$. To see this, first let us prove the following lemma.

Lemma L.11. Suppose $x \in \mathbf{R}$ and $n \in \mathbf{N}$. There exists a $\delta > 0$ such that if $\frac{a}{b} \neq x$ is a rational number contained in $V_\delta(x)$ with $b > 0$, then $b > n$.

Proof. Suppose $b \in \mathbf{N}$ is such that $1 \leq b \leq n$. Since $I := [x-1, x+1]$ is an interval of length 2, there are either $2b$ or $2b+1$ rationals of the form $\frac{a}{b}$ contained in I. (To fit the most rationals of this form inside I, we should place the first such rational $\frac{a}{b}$ on the left endpoint $x - 1$; then $\frac{a+2b}{b} = \frac{a}{b} + 2 = x + 1$ is the right endpoint. Thus we have the $2b + 1$ rational numbers $\frac{a}{b}, \frac{a+1}{b}, \ldots, \frac{a+2b}{b}$ contained in I. In the general case, the left endpoint will not be of the form $\frac{a}{b}$ and so there will be only $2b$ rationals of this form contained in I.) Given this, the set

$$A = \left\{ \left| x - \tfrac{a}{b} \right| : \tfrac{a}{b} \in I \setminus \{x\},\, 1 \le b \le n \right\}$$

is non-empty and finite, so that $\delta := \min A$ exists (Lemma L.3); notice that $\delta > 0$ since each element of A is strictly positive. It follows that $V_\delta(x)$ can contain only rationals $\frac{a}{b}$ with denominators $b > n$, other than possibly x itself. □

Now we can prove that $\lim_{x\to 1} t(x) = 0$. Let $\varepsilon > 0$ be given and let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \varepsilon$. By Lemma L.11, there exists a $\delta > 0$ such that if $\frac{a}{b} \neq 1$ is a rational number contained in $V_\delta(1)$, then $b > n$. Suppose $x \in V_\delta(1)$. If x is irrational then $t(x) = 0 \in V_\varepsilon(0)$, and if $x = \frac{a}{b} \neq 1$ is rational then

$$0 \le t(x) = \frac{1}{b} < \frac{1}{n} < \varepsilon \quad \Rightarrow \quad t(x) \in V_\varepsilon(0).$$

In either case, $x \in V_\delta(1) \setminus \{1\}$ implies that $t(x) \in V_\varepsilon(0)$ and thus $\lim_{x\to 1} t(x) = 0$.

Exercise 4.2.4. Consider the reasonable but erroneous claim that

$$\lim_{x\to 10} 1/[[x]] = 1/10.$$

(a) Find the largest δ that represents a proper response to the challenge of $\varepsilon = 1/2$.

(b) Find the largest δ that represents a proper response to $\varepsilon = 1/50$.

(c) Find the largest ε challenge for which there is no suitable δ response possible.

Solution. Let $f(x) = \frac{1}{[[x]]}$, which is defined provided $[[x]] \neq 0$, which is the case if and only if $x < 0$ or $x \ge 1$. Thus the domain of f is $A = (-\infty, 0) \cup [1, \infty)$.

(a) Let $\delta = 8$ and observe that

$$x \in V_\delta(10) = (2, 18) \quad \Rightarrow \quad f(x) \in \left[\frac{1}{17}, \frac{1}{2}\right] \subseteq \left(-\frac{2}{5}, \frac{3}{5}\right) = V_{1/2}\left(\frac{1}{10}\right).$$

Thus $\delta = 8$ is a valid response to the challenge of $\varepsilon = \frac{1}{2}$. If $\delta > 8$, then there exists an $x \in V_\delta(10)$ such that $1 \le x < 2$, which gives $f(x) = 1 \notin \left(-\frac{2}{5}, \frac{3}{5}\right) = V_{1/2}\left(\frac{1}{10}\right)$. Hence $\delta = 8$ is the largest proper response to the challenge of $\varepsilon = \frac{1}{2}$.

(b) Let $\delta = 1$ and observe that

$$x \in V_\delta(10) = (9, 11) \quad \Rightarrow \quad f(x) \in \left[\frac{1}{10}, \frac{1}{9}\right] \subseteq \left(\frac{2}{25}, \frac{3}{25}\right) = V_{1/50}\left(\frac{1}{10}\right).$$

Thus $\delta = 1$ is a valid response to the challenge of $\varepsilon = \frac{1}{50}$. If $\delta > 1$, then there exists an $x \in V_\delta(10)$ such that $8 \le x < 9$, which gives $f(x) = \frac{1}{8} \notin \left(\frac{2}{25}, \frac{3}{25}\right) = V_{1/50}\left(\frac{1}{10}\right)$. Hence $\delta = 1$ is the largest proper response to the challenge of $\varepsilon = \frac{1}{50}$.

(c) Suppose that $\varepsilon = \frac{1}{90}$ and $\delta > 0$. Notice that there exists an $x \in V_\delta(10)$ such that $9 \le x < 10$, which gives $f(x) = \frac{1}{9} \notin \left(\frac{4}{45}, \frac{1}{9}\right) = V_\varepsilon\left(\frac{1}{10}\right)$. Thus there is no valid δ response to the challenge of $\varepsilon = \frac{1}{90}$.

Now suppose $\varepsilon > \frac{1}{90}$, let $\delta = 1$, and observe that

$$x \in V_\delta(10) = (9, 11) \quad \Rightarrow \quad f(x) \in \left[\frac{1}{10}, \frac{1}{9}\right] \subseteq V_\varepsilon\left(\frac{1}{10}\right).$$

Thus $\delta = 1$ is a valid response to this ε challenge.

We may conclude that $\varepsilon = \frac{1}{90}$ is the largest challenge for which there is no suitable δ response possible.

Exercise 4.2.5. Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x + 4) = 10$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2}(x^2 + x - 1) = 5$.

(d) $\lim_{x\to 3} 1/x = 1/3$.

Solution.

(a) Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{3}$. If $x \in \mathbf{R}$ is such that $0 < |x - 2| < \delta$, then

$$|(3x + 4) - 10| = |3x - 6| = 3|x - 2| < 3\delta = \varepsilon.$$

Thus $\lim_{x\to 2}(3x + 4) = 10$.

(b) Let $\varepsilon > 0$ be given and let $\delta = \varepsilon^{1/3}$. If $x \in \mathbf{R}$ is such that $0 < |x| < \delta$, then

$$\left|x^3\right| = |x|^3 < \delta^3 = \varepsilon.$$

Thus $\lim_{x\to 0} x^3 = 0$.

(c) Let $\varepsilon > 0$ be given. Observe that if $|x - 2| < 1$, i.e. $x \in (1, 3)$, then $x + 3 \in (4, 7)$. Let $\delta = \min\left\{\frac{\varepsilon}{7}, 1\right\}$. If $x \in \mathbf{R}$ is such that $0 < |x - 2| < \delta$, then $x \in (1, 3)$ and thus

$$\left|x^2 + x - 1 - 5\right| = |x + 3||x - 2| < 7\delta \le \varepsilon.$$

Thus $\lim_{x\to 2}(x^2 + x - 1) = 5$.

(d) Let $\varepsilon > 0$ be given. Observev that if $|x - 3| < 1$, i.e. $x \in (2, 4)$, then $\frac{1}{3x} \in \left(\frac{1}{12}, \frac{1}{6}\right)$. Let $\delta = \min\{6\varepsilon, 1\}$ and note that if $x \in \mathbf{R} \setminus \{0\}$ is such that $0 < |x - 3| < \delta$, then $x \in (2, 4)$ and thus

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|x - 3|}{|3x|} < \frac{\delta}{6} \le \varepsilon.$$

Thus $\lim_{x\to 3} 1/x = 1/3$.

Exercise 4.2.6. Decide if the following claims are true or false, and give short justifications of each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.

(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.

(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x)-2]^2 = 3(L-2)^2$.

(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f.)

Solution.

(a) This is true, since if $0 < \delta' < \delta$ then $V_{\delta'}(c) \subseteq V_\delta(c)$ for any $c \in \mathbf{R}$.

(b) This is false. For a counterexample, consider Thomae's function t. In Exercise 4.2.3 we showed that $\lim_{x\to 1} t(x) = 0$, but $t(1) = 1$.

(c) This is true and follows from several applications of the Algebraic Limit Theorem for Functional Limits (Corollary 4.2.4).

(d) This is false. Define $f, g : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ by $f(x) = x$ and $g(x) = \frac{1}{x}$. It is straightforward to verify that $\lim_{x\to 0} f(x) = 0$, but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} 1 = 1$.

Exercise 4.2.7. Let $g : A \to \mathbf{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \le M$ for all $x \in A$.

Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Solution. Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} g(x) = 0$, there is a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in A$ implies that $|g(x)| < \frac{\varepsilon}{M}$. Observe that for such x we have

$$|f(x)g(x)| = |f(x)||g(x)| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Thus $\lim_{x\to c} g(x)f(x) = 0$.

Exercise 4.2.8. Compute each limit or state that it does not exist. Use the tools developed in this section to justify each conclusion.

(a) $\lim_{x\to 2} \frac{|x-2|}{x-2}$

(b) $\lim_{x\to 7/4} \frac{|x-2|}{x-2}$

(c) $\lim_{x\to 0} (-1)^{[[1/x]]}$

(d) $\lim_{x\to 0} \sqrt[3]{x}(-1)^{[[1/x]]}$

Solution.

(a) Let $f : \mathbf{R} \setminus \{2\} \to \mathbf{R}$ be given by $f(x) = \frac{|x-2|}{x-2}$. Observe that

$$f(x)=\begin{cases}1 & \text{if } x>2,\\ -1 & \text{if } x<2.\end{cases}$$

We claim that $\lim_{x\to 2} f(x)$ does not exist. To see this, consider the sequences (x_n) and (y_n) given by $x_n = 2+\frac{1}{n}$ and $y_n = 2-\frac{1}{n}$, which satisfy $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 2$. However,

$$\lim_{n\to\infty} f(x_n) = 1 \neq -1 \lim_{n\to\infty} f(y_n).$$

Our claim now follows from Corollary 4.2.5.

(b) Define f as in part (a). We claim that $\lim_{x\to 7/4} f(x) = -1$. To see this, let $\varepsilon > 0$ be given. If $x \in \mathbf{R}\setminus\{2\}$ is such that $0 < \left|x-\frac{7}{4}\right| < \frac{1}{4}$, i.e. $x \in \left(\frac{3}{2}, 2\right)$, then

$$|f(x)-(-1)| = (-1+1) = 0 < \varepsilon.$$

Thus $\lim_{x\to 7/4} f(x) = -1$.

(c) We claim that $\lim_{x\to 0} (-1)^{[[1/x]]}$ does not exist. To see this, consider the sequences (x_n) and (y_n) given by $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{2n+1}$, which satisfy $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. However,

$$\lim_{n\to\infty} (-1)^{[[1/x_n]]} = \lim_{n\to\infty} (-1)^{[[2n]]} = 1 \neq -1 = \lim_{n\to\infty} (-1)^{[[2n+1]]} = \lim_{n\to\infty} (-1)^{[[1/y_n]]}.$$

Our claim now follows from Corollary 4.2.5.

(d) Let $\varepsilon > 0$ be given and let $\delta = \varepsilon^3$. If $x \in \mathbf{R}$ is such that $0 < |x| < \delta$, then

$$\left|\sqrt[3]{x}\right| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \varepsilon.$$

Thus $\lim_{x\to 0} \sqrt[3]{x} = 0$. Since the function $(-1)^{[[1/x]]}$ is evidently bounded, we may apply Exercise 4.2.7 to conclude that $\lim_{x\to 0} \sqrt[3]{x}(-1)^{[[1/x]]} = 0$.

Exercise 4.2.9 (Infinite Limits). The statement $\lim_{x\to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construct a rigorous definition in the challenge-response style of Definition 4.2.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\varepsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:

Definition: $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x-c| < \delta$, it follows that $f(x) > M$.

(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

(b) Now construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} 1/x = 0$.

(c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ look like? Give an example of such a limit.

Solution.

(a) Let $M > 0$ be given and let $\delta = \frac{1}{\sqrt{M}} > 0$. If $x \in \mathbf{R}$ is such that $0 < |x| < \delta$, then observe that

$$\frac{1}{|x|} > \sqrt{M} > 0 \quad \Rightarrow \quad \frac{1}{x^2} > M.$$

It follows that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(b) The statement $\lim_{x\to\infty} f(x) = L$ means that for all $\varepsilon > 0$ we can find an $M > 0$ such that whenever $x > M$, it follows that $|f(x) - L| < \varepsilon$.

To see that $\lim_{x\to\infty} \frac{1}{x} = 0$, let $\varepsilon > 0$ be given, let $M = \frac{1}{\varepsilon}$, and observe that

$$x > M = \frac{1}{\varepsilon} \quad \Rightarrow \quad \frac{1}{x} < \varepsilon.$$

(c) The statement $\lim_{x\to\infty} f(x) = \infty$ means that for all $M > 0$ we can find a $K > 0$ such that whenever $x > K$, it follows that $f(x) > M$; it is straightforward to verify that $\lim_{x\to\infty} x = \infty$, for example.

Exercise 4.2.10 (Right and Left Limits). Introductory calculus courses typically refer to the *right-hand limit* of a function as the limit obtained by "letting x approach a from the right-hand side."

(a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \quad \text{and} \quad \lim_{x\to a^-} f(x) = M.$$

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Solution.

(a) Suppose we have a function $f : A \to \mathbf{R}$ and $a \in \mathbf{R}$ is a limit point of $A \cap (a, \infty)$. We say that $\lim_{x\to a^+} f(x) = L$ provided that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $a < x < a + \delta$ and $x \in A$, it follows that $|f(x) - L| < \varepsilon$. Similarly, if $a \in \mathbf{R}$ is a limit point of $A \cap (-\infty, a)$, we say that $\lim_{x\to a^-} f(x) = M$ provided that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $a - \delta < x < a$ and $x \in A$, it follows that $|f(x) - M| < \varepsilon$.

(b) Let $f : A \to \mathbf{R}$ and $a \in A$ be given, and suppose that a is a limit point of both $A \cap (a, \infty)$ and $A \cap (-\infty, a)$.

If $\lim_{x\to a} f(x) = L$ then certainly $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, since for any $x \in A$ both of the statements $a < x < a + \delta$ and $a - \delta < x < a$ imply that $0 < |x - a| < \delta$. Suppose therefore that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ and let $\varepsilon > 0$ be given. There are positive real numbers δ_1 and δ_2 such that

$$a < x < a + \delta_1 \text{ and } x \in A \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

$$\text{and} \qquad a - \delta_2 < x < a \text{ and } x \in A \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $x \in A$ is such that $0 < |x - a| < \delta$, then either

$$a < x < a + \delta \leq a + \delta_1 \quad \Rightarrow \quad |f(x) - L| < \varepsilon, \text{ or}$$

$$a - \delta_2 \leq a - \delta < x < a \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

In either case we have $|f(x) - L| < \varepsilon$. Thus $\lim_{x \to a} f(x) = L$.

Exercise 4.2.11 (Squeeze Theorem). Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point c of A, show $\lim_{x \to c} g(x) = L$.

Solution. Suppose (x_n) is a sequence contained in A satisfying $x_n \neq c$ and $\lim_{n \to \infty} x_n = c$. By assumption we have $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbf{N}$, and Theorem 4.2.3 guarantees that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} h(x_n) = L$. We may now apply the Squeeze Theorem for sequences (Exercise 2.3.3) to see that $\lim_{n \to \infty} g(x_n) = L$, and Theorem 4.2.3 then allows us to conclude that $\lim_{x \to c} g(x) = L$.

4.3. Continuous Functions

Exercise 4.3.1. Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

(b) Prove that g is continuous at a point $c \neq 0$. (The identity

$$a^3 - b^3 = (a - b)(a^2 + ab + b)$$

will be helpful.)

Solution.

(a) Let $\varepsilon > 0$ be given and let $\delta = \varepsilon^3$. If $x \in \mathbf{R}$ is such that $|x| < \delta$, then

$$\left|\sqrt[3]{x}\right| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \varepsilon.$$

Thus g is continuous at $c = 0$.

(b) Taking $a = x^{1/3}$ and $b = c^{1/3}$ in the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ gives

$$x - c = \left(x^{1/3} - c^{1/3}\right)\left(x^{2/3} + (xc)^{1/3} + c^{2/3}\right)$$
$$\implies \quad |x - c| = \left|x^{1/3} - c^{1/3}\right|\left|x^{2/3} + (xc)^{1/3} + c^{2/3}\right|.$$

If we take x close enough to c so that x and c have the same sign, i.e. take x such that $|x - c| < |c|$, then $xc > 0$ and so

$$\left|x^{2/3} + (xc)^{1/3} + c^{2/3}\right| = x^{2/3} + (xc)^{1/3} + c^{2/3} \geq c^{2/3}.$$

Let $\delta = \min\left\{|c|, c^{2/3}\varepsilon\right\}$ and suppose $x \in \mathbf{R}$ is such that $|x - c| < \delta$. By the previous discussion, we then have

$$\left|x^{1/3} - c^{1/3}\right| \leq \frac{|x - c|}{c^{2/3}} \leq \varepsilon.$$

Thus g is continuous at c.

Exercise 4.3.2. To gain a deeper understanding of the relationship between ε and δ in the definition of continuity, let's explore some modest variations of Definition 4.3.1. In all of these, let f be a function defined on all of $\mathbf{R}$.

(a) Let's say f is *onetinuous* at c if for all $\varepsilon > 0$ we can choose $\delta = 1$ and it follows that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuous on all of $\mathbf{R}$.

(b) Let's say f is *equaltinuous* at c if for all $\varepsilon > 0$ we can choose $\delta = \varepsilon$ and it follows that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on $\mathbf{R}$ that is nowhere onetinuous, or explain why there is no such function.

(c) Let's say f is *lesstinuous* at c if for all $\varepsilon > 0$ we can choose $0 < \delta < \varepsilon$ and it follows that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find an example of a function that is lesstinuous on $\mathbf{R}$ that is nowhere equaltinuous, or explain why there is no such function.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

Solution.

(a) Let f be given by $f(x) = 0$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\varepsilon > 0$ be given. If $x \in \mathbf{R}$ is such that $|x - c| < 1$, then

$$|f(x) - f(c)| = |0 - 0| = 0 < \varepsilon.$$

Thus f is onetinuous on $\mathbf{R}$.

(b) Let f be given by $f(x) = x$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\varepsilon > 0$ be given. If $x \in \mathbf{R}$ is such that $|x - c| < \varepsilon$, then

$$|f(x) - f(c)| = |x - c| < \varepsilon.$$

Thus f is equaltinuous on $\mathbf{R}$. However, f is nowhere onetinuous. Fix $c \in \mathbf{R}$ again and consider $\varepsilon = \frac{1}{4}$. Note that $x = c + \frac{1}{2}$ satisfies $|x - c| = \frac{1}{2} < 1$, however

$$|f(x) - f(c)| = |x - c| = \frac{1}{2} > \frac{1}{4} = \varepsilon.$$

Thus f is nowhere onetinuous.

(c) Let f be given by $f(x) = 2x$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ let $\varepsilon > 0$ be given, and let $\delta = \frac{\varepsilon}{2} < \varepsilon$. If $x \in \mathbf{R}$ is such that $|x - c| < \delta$, then

$$|f(x) - f(c)| = 2|x - c| < 2\delta = \varepsilon.$$

Thus f is lesstinuous on $\mathbf{R}$. However, f is nowhere equaltinuous. Fix $c \in \mathbf{R}$ again and let $\varepsilon = 1$. Note that $x = c + \frac{3}{4}$ satisfies $|x - c| = \frac{3}{4} < \varepsilon$, however

$$|f(x) - f(c)| = 2|x - c| = \frac{3}{2} > \varepsilon.$$

Thus f is nowhere equaltinuous.

(d) It is clear that every lesstinuous function is continuous. We claim that every continuous function is lesstinuous. To see this, let f be a continuous function. Fix $c \in \mathbf{R}$ and $\varepsilon > 0$. Since f is continuous at c, there is a $\delta' > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta'$. Let $\delta = \min\{\delta', \frac{\varepsilon}{2}\}$, so that $0 < \delta < \varepsilon$, and observe that if $x \in \mathbf{R}$ is such that $|x - c| < \delta$ then x also satisfies $|x - c| < \delta'$, whence $|f(x) - f(c)| < \varepsilon$.

Exercise 4.3.3.

(a) Supply a proof for Theorem 4.3.9 using the ε-δ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Solution.

(a) Let $a \in A$ and $\varepsilon > 0$ be given. By assumption we have $f(a) \in B$, so g is continuous at $f(a)$. There then exists a $\delta_1 > 0$ such that

$$|y - f(a)| < \delta_1 \text{ and } y \in B \quad \Rightarrow \quad |g(y) - g(f(a))| < \varepsilon. \tag{1}$$

Since f is continuous at a, there exists a $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \text{ and } x \in A \quad \Rightarrow \quad |f(x) - f(a)| < \delta_1. \tag{2}$$

Combining (1) and (2) shows that

$$\begin{aligned} |x - a| < \delta_2 \text{ and } x \in A \quad &\Rightarrow \quad |f(x) - f(a)| < \delta_1 \text{ and } f(x) \in B \\ &\Rightarrow \quad |g(f(x)) - g(f(a))| < \varepsilon. \end{aligned}$$

Thus $g \circ f$ is continuous at a.

(b) Let $a \in A$ be given and suppose $(a_n)_{n=1}^{\infty}$ is contained in A and satisfies $\lim_{n\to\infty} a_n = a$. Since f is continuous at a, Theorem 4.3.2 (iii) gives us $\lim_{n\to\infty} f(a_n) = f(a)$. By assumption g is continuous at $f(a) \in B$ and $(f(a_n))_{n=1}^{\infty}$ is contained in B, so Theorem 4.3.2 (iii) gives us $\lim_{n\to\infty} g(f(a_n)) = g(f(a))$. One more application of Theorem 4.3.2 (iii) allows us to conclude that $g \circ f$ is continuous at a.

Exercise 4.3.4. Assume f and g are defined on all of $\mathbf{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$.

(a) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.

(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Solution.

(a) Let f be given by $f(x) = 0$ for all $x \in \mathbf{R}$ and let g be given by

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$, however note that $g(f(x)) = g(0) = 1$ for all $x \in \mathbf{R}$. It follows that

$$\lim_{x \to 0} g(f(x)) = 1 \neq 0.$$

(b) By Theorem 4.3.9 the composition $g \circ f$ is continuous. Since f and g are defined on all of $\mathbf{R}$, Theorem 4.3.2 (iv) lets us write

$$\lim_{x \to p} g(f(x)) = g(f(p)) = g\left(\lim_{x \to p} f(x)\right) = g(q) = \lim_{x \to q} g(x).$$

(c) As the counterexample in part (a) shows, the result does not hold if we only assume that f is continuous. However, it does hold if we assume that g is continuous. To see this, let (x_n) be some sequence satisfying $\lim_{n \to \infty} x_n = p$ and $x_n \neq p$. Theorem 4.2.3 shows that $\lim_{n \to \infty} f(x_n) = q$, and since g is continuous the sequential characterization of continuity (Theorem 4.3.2 (iii)) implies that

$$\lim_{n \to \infty} g(f(x_n)) = g(q) = r,$$

where the last equality also follows from the continuity of g. Theorem 4.2.3 allows us to conclude that $\lim_{x \to p} g(f(x)) = r$.

Exercise 4.3.5. Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \to \mathbf{R}$ is continuous at c.

Solution. Since c is an isolated point of A, there exists a $\delta > 0$ such that $V_\delta(c) \cap A = \{c\}$. Let $\varepsilon > 0$ be given. If $x \in A$ is such that $|x - c| < \delta$ then it must be the case that $x = c$. It follows that

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon.$$

Thus f is continuous at c.

Exercise 4.3.6. Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 such that $f(x)g(x)$ and $f(x) = g(x)$ are continuous at 0.

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

(d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.

(e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

Solution.

(a) Let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Neither f nor g is continuous at 0, however note that for all $x \in \mathbf{R}$ we have

$$f(x)g(x) = 0 \quad \text{and} \quad f(x) + g(x) = 1.$$

Thus fg and $f + g$ are continuous at 0.

(b) This is impossible. If f and $f + g$ are continuous at 0 then Theorem 4.3.4 implies that $g = f + g - f$ is continuous at 0.

(c) If we take g as in part (a) and let $f(x) = 0$ for all $x \in \mathbf{R}$, then g is not continuous at 0 but $f = fg$ is continuous at 0.

(d) Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} \sqrt{2} - 1 & \text{if } x \neq 0, \\ \sqrt{2} + 1 & \text{if } x = 0, \end{cases}$$

and note that f is discontinuous at 0. Note further that $f(x) + \frac{1}{f(x)} = 2\sqrt{2}$ for all $x \in \mathbf{R}$. It follows that $f + \frac{1}{f}$ is continuous at 0.

(e) This is impossible. As we showed in Exercise 4.3.1, the function $x \mapsto \sqrt[3]{x}$ is continuous everywhere. It follows that if $[f(x)]^3$ is continuous at 0 then, by Theorem 4.3.9, the composition

$$f(x) = \left([f(x)]^3\right)^{1/3}$$

must also be continuous at 0.

Exercise 4.3.7.

(a) Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on $\mathbf{R}$.

(b) Review the definition of Thomae's function in Section 4.1 and demonstrate that it fails to be continuous at every rational point.

(c) Use the characterization of continuity in Theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in $\mathbf{R}$. (Given $\varepsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \varepsilon\}$.)

Solution.

(a) Let $g : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function, i.e.

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Suppose $c \in \mathbf{Q}$. By the density of $\mathbf{I}$ in $\mathbf{R}$, for any $\delta > 0$ there is an irrational number $x \in \mathbf{I}$ such that $x \in V_\delta(c)$; it follows that $g(x) = 0 \notin V_1(1) = V_1(g(c))$. Thus, by Theorem 4.3.2 (ii), g is not continuous at c.

Similarly, suppose $c \in \mathbf{I}$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, for any $\delta > 0$ there is a rational number $x \in \mathbf{Q}$ such that $x \in V_\delta(c)$; it follows that $g(x) = 1 \notin V_1(0) = V_1(g(c))$. Thus, by Theorem 4.3.2 (ii), g is not continuous at c.

We have now shown that g fails to be continuous at each $c \in \mathbf{R}$, i.e. g is nowhere-continuous on $\mathbf{R}$.

(b) Let $t : \mathbf{R} \to \mathbf{R}$ be Thomae's function, i.e.

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Suppose $c \in \mathbf{Q}$. The density of $\mathbf{I}$ in $\mathbf{R}$ allows us to pick a sequence of irrational number (x_n) such that $\lim_{n\to\infty} x_n = c$. We then have $t(x_n) = 0$ for each $n \in \mathbf{N}$ and so $\lim_{n\to\infty} t(x_n) = 0$. However, $t(c)$ is strictly positive; it follows that $\lim_{n\to\infty} t(x_n) \neq t(c)$ and hence, by Corollary 4.3.3, t is not continuous at $c \in \mathbf{Q}$. Thus t fails to be continuous on $\mathbf{Q}$.

(c) Suppose $c \in \mathbf{I}$ and suppose we have a sequence (x_n) such that $\lim_{n\to\infty} x_n = c$. Our aim is to show that $\lim_{n\to\infty} t(x_n) = t(c) = 0$. Let $\varepsilon > 0$ be given and choose $K \in \mathbf{N}$ such that $\frac{1}{K} < \varepsilon$. By Lemma L.11, there exists a $\delta > 0$ such that if $y = \frac{a}{b}$ is a rational number contained in $V_\delta(c)$ with $b > 0$, then $b > K$. For such y we have $t(y) = \frac{1}{b} < \frac{1}{K} < \varepsilon$. Since $\lim_{n\to\infty} x_n = c$, there is an $N \in \mathbf{N}$ such that $x_n \in V_\delta(c)$ for all $n \geq N$. Suppose $n \in \mathbf{N}$ satisfies $n \geq N$. There are two cases.

Case 1. If $x_n \in \mathbf{I}$, then $|t(x_n)| = 0 < \varepsilon$.

Case 2. If $x_n \in \mathbf{Q}$, then since $x_n \in V_\delta(c)$ we have $|t(x_n)| < \frac{1}{K} < \varepsilon$.

In either case we have $|t(x_n)| < \varepsilon$ and thus $\lim_{n\to\infty} t(x_n) = t(c) = 0$, as desired. Theorem 4.3.2 (iii) allows us to conclude that t is continuous at each $c \in \mathbf{I}$.

Exercise 4.3.8. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbf{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(b) If $g(r) = 0$ for all $r \in \mathbf{Q}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.

(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbf{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution.

(a) This is true. Let (x_n) be the sequence given by $x_n = 1 - \frac{1}{n}$. Since g is continuous at 1 and $\lim_{n\to\infty} x_n = 1$, Theorem 4.3.2 (iii) implies that $\lim_{n\to\infty} g(x_n) = g(1)$. Note that $x_n < 1$ for each $n \in \mathbf{N}$ and thus $g(x_n) \geq 0$ for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) allows us to conclude that $\lim_{n\to\infty} g(x_n) = g(1) \geq 0$ also.

(b) This is true. Let $x \in \mathbf{R}$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there is a sequence (r_n) of rational numbers such that $\lim_{n\to\infty} r_n = x$. On one hand, by the continuity of g at x, we must have $\lim_{n\to\infty} g(r_n) = g(x)$ (Theorem 4.3.2 (iii)). On the other hand, $g(r_n) = 0$ for all $n \in \mathbf{N}$ and thus $\lim_{n\to\infty} g(r_n) = 0$. Since the limit of a sequence is unique (Theorem 2.2.7), it follows that $g(x) = 0$.

(c) This is true. Since g is continuous at x_0, for $\varepsilon = g(x_0) > 0$ there is a $\delta > 0$ such that $g(x) \in V_\varepsilon(g(x_0)) = (0, 2g(x_0))$ whenever $x \in V_\delta(x_0)$. In other words, for each of the uncountably many $x \in (x_0 - \delta, x_0 + \delta)$ we have $g(x) > 0$.

Exercise 4.3.9. Assume $h : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution. Suppose that (x_n) is a convergent sequence contained in K with $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$. On one hand, the continuity of h implies that $\lim_{n\to\infty} h(x_n) = h(x)$. On the other hand, because each $x_n \in K$, we have $h(x_n) = 0$ for each $n \in \mathbf{N}$ and thus $\lim_{n\to\infty} h(x_n) = 0$. The uniqueness of the limit of a sequence (Theorem 2.2.7) now implies that $h(x) = 0$, i.e. $x \in K$. Theorem 3.2.8 allows us to conclude that K is closed.

Exercise 4.3.10. Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a+b) + |a-b|].$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.

(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbf{N}$, define f_n on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n, \\ n|x| & \text{if } |x| < 1/n. \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$.

Solution.

(a) First, let us show that the function $x \mapsto |x|$ is continuous on $\mathbf{R}$. If $y \in \mathbf{R}$ and $\varepsilon > 0$, let $\delta = \varepsilon$ and suppose that $|x - y| < \delta$. The reverse triangle inequality (Exercise 1.2.6 (d)) shows that

$$||x| - |y|| \leq |x - y| < \delta = \varepsilon.$$

Thus $x \mapsto |x|$ is continuous on $\mathbf{R}$.

Now suppose that $f_1, f_2 : A \to \mathbf{R}$ are two continuous functions defined on some domain $A \subseteq \mathbf{R}$. For any $x \in A$, note that

$$g(x) = \max\{f_1(x), f_2(x)\} = \tfrac{1}{2}[(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|].$$

Since f_1 and f_2 are continuous on A, and we showed that $x \mapsto |x|$ is continuous everywhere, Theorem 4.3.9 and several applications of Theorem 4.3.4 show that g is also continuous on A.

Using the observation that

$$\max\{f_1(x), f_2(x), \ldots, f_n(x)\} = \max\{\max\{f_1(x), f_2(x), \ldots, f_{n-1}(x)\}, f_n(x)\},$$

a straightforward induction argument on n (the base case was handled in the previous paragraph) shows that the maximum of n continuous functions is a continuous function.

(b) If $x = 0$ then for each $n \in \mathbf{N}$ we have $f_n(0) = 0$ and thus $h(0) = 0$. If $x \neq 0$ then choose $n \in \mathbf{N}$ such that $\frac{1}{n} < |x|$ and notice that $f_n(x) = 1$, so that $h(x) = 1$. Thus h is the function

$$h(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is not continuous at 0.

Exercise 4.3.11 (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbf{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbf{R}$.

(a) Show that f is continuous on $\mathbf{R}$.

(b) Pick some point $y_1 \in \mathbf{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e. $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is *any* arbitrary point in $\mathbf{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined by (b).

Solution.

(a) Let $y \in \mathbf{R}$ and $\varepsilon > 0$ be given. Let $\delta = c^{-1}\varepsilon$, suppose that $x \in \mathbf{R}$ is such that $|x - y| < \delta$, and observe that

$$|f(x) - f(y)| \leq c|x - y| < c\delta = \varepsilon.$$

Thus f is continuous at each $y \in \mathbf{R}$.

(b) Suppose $n > m$ are positive integers. Repeatedly applying the triangle inequality yields

$$|y_n - y_m| \leq |y_n - y_{n-1}| + \cdots + |y_{m+1} - y_m| = \sum_{k=m}^{n-1} |y_{k+1} - y_k|.$$

Now we use the hypothesis that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbf{R}$ and the definition of the sequence $y_n = f(y_{n-1})$ to see that

$$\sum_{k=m}^{n-1} |y_{k+1} - y_k| \leq \sum_{k=m}^{n-1} c|y_k - y_{k-1}| \leq \cdots \leq \sum_{k=m}^{n-1} c^{k-1}|y_2 - y_1| = c^{-2}|y_2 - y_1| \sum_{k=m+1}^{n} c^k.$$

If we let $s_n = \sum_{k=0}^{n} c^k$, then we have shown that for all positive integers $n > m$ we have the inequality

$$|y_n - y_m| \leq c^{-2}|y_2 - y_1|(s_n - s_m). \tag{1}$$

Let $\varepsilon > 0$ be given. The series $\sum_{k=0}^{\infty} c^k$ is convergent since $0 < c < 1$, so the sequence (s_n) is Cauchy. Thus there exists an $N \in \mathbf{N}$ such that

$$n > m \geq N \quad \Rightarrow \quad |s_n - s_m| = s_n - s_m < \frac{c^2}{|y_2 - y_1| + 1}\varepsilon. \tag{2}$$

Suppose n, m are positive integers such that $n > m \geq N$. It follows from (1) and (2) that

$$|y_n - y_m| \leq c^{-2}|y_2 - y_1| \frac{c^2}{|y_2 - y_1| + 1} \varepsilon < \varepsilon.$$

Thus (y_n) is a Cauchy sequence.

(c) Since f is continuous at y (by part (a)), we have $\lim_{n\to\infty} f(y_n) = f(y)$. It follows that

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} f(y_n) = f(y).$$

For uniqueness, observe that for any $x \in \mathbf{R}$ such that $x = f(x)$ we have

$$|x - y| = |f(x) - f(y)| \leq c|x - y|.$$

If $|x - y|$ were not zero this would imply that $c \geq 1$. Since $0 < c < 1$, it must be the case that $|x - y| = 0$, i.e. $x = y$.

(d) Let $x_1 = x$ and $x_{n+1} = f(x_n)$. As we just proved, (x_n) converges to some $y' \in \mathbf{R}$ such that $f(y') = y'$. The uniqueness part of (c) then implies that $y' = y$.

Exercise 4.3.12. Let $F \subseteq \mathbf{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbf{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Solution. If A and B are non-empty and bounded below subsets of $\mathbf{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$, then it is straightforward to verify that $\inf A \leq \inf B$. Fix $c \in \mathbf{R}$ and note that for any $x \in \mathbf{R}$ and $a \in F$ we have $|x - a| \leq |x - c| + |c - a|$. It follows that

$$\inf\{|x - a| : a \in F\} \leq \inf\{|x - c| + |c - a| : a \in F\}.$$

A statement analogous to Example 1.3.7 for infima then gives us

$$\inf\{|x - a| : a \in F\} \leq |x - c| + \inf\{|c - a| : a \in F\}$$

i.e. $g(x) - g(c) \leq |x - c|$. We can similarly derive $g(c) - g(x) \leq |x - c|$ and hence

$$|g(x) - g(c)| \leq |x - c|.$$

Thus for any $\varepsilon > 0$ we can take $\delta = \varepsilon$ and obtain

$$|x - c| < \delta \quad \Rightarrow \quad |g(x) - g(c)| < \varepsilon.$$

It follows that g is continuous at each $c \in \mathbf{R}$.

Suppose that $g(x) = 0$. Using Exercise 1.3.1 (b) we can choose a sequence (a_n) contained in F satisfying $\lim_{n\to\infty}|x - a_n| = g(x) = 0$, which is equivalent to $\lim_{n\to\infty} a_n = x$. Since F is closed, Theorem 3.2.8 then implies that $x \in F$. Thus if $x \notin F$ then it must be the case that $g(x) \neq 0$.

Exercise 4.3.13. Let f be a function defined on all of $\mathbf{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbf{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbf{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbf{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbf{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution.

(a) We have $f(0) = f(0+0) = f(0) + f(0)$ and thus $f(0) = 0$. Furthermore, for any $x \in \mathbf{R}$,

$$0 = f(0) = f(x-x) = f(x) + f(-x) \quad \Rightarrow \quad f(-x) = -f(x).$$

(b) We will show that $f(n) = kn$ for all $n \in \mathbf{N}$ by induction on n. The base case is clear, so suppose that $f(n) = kn$ for some $n \in \mathbf{N}$ and observe that

$$f(n+1) = f(n) + f(1) = kn + k = k(n+1).$$

This completes the induction step and the proof.

Combining the identity $f(n) = kn$ with $f(-x) = -f(x)$ from part (a) shows that $f(z) = kz$ for all $z \in \mathbf{Z}$.

Now suppose that $r = \frac{m}{n}$ is a rational number. On one hand, using what we just proved,

$$f\Big(n \cdot \frac{m}{n}\Big) = f(m) = km.$$

On the other hand, using the additivity of f,

$$f\Big(n \cdot \frac{m}{n}\Big) = f\left(\sum_{j=1}^{n} \frac{m}{n}\right) = \sum_{j=1}^{n} f\Big(\frac{m}{n}\Big) = nf\Big(\frac{m}{n}\Big).$$

Thus $nf\big(\frac{m}{n}\big) = km$, i.e. $f(r) = kr$.

(c) Let $c \in \mathbf{R}$ be given and suppose (x_n) is a sequence satisfying $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty}(x_n - c) = 0$ and f is continuous at 0, we must have

$$\lim_{n\to\infty} f(x_n - c) = f(0) = 0.$$

The additivity of f shows that $f(x_n - c) = f(x_n) - f(c)$ for each $n \in \mathbf{N}$. It follows that

$$0 = \lim_{n\to\infty} f(x_n - c) = \lim_{n\to\infty} (f(x_n) - f(c)) = \Big(\lim_{n\to\infty} f(x_n)\Big) - f(c),$$

which implies $\lim_{n\to\infty} f(x_n) = f(c)$. Thus f is continuous at each $c \in \mathbf{R}$.

By Theorem 4.3.4, the function $f(x) - kx$ is continuous on all of $\mathbf{R}$ and, by part (b), satisfies $f(r) - kr = 0$ for each $r \in \mathbf{Q}$. Exercise 4.3.8 (b) allows us to conclude that $f(x) - kx = 0$, i.e. $f(x) = kx$, for all $x \in \mathbf{R}$.

Exercise 4.3.14.

(a) Let F be a closed set. Construct a function $f : \mathbf{R} \to \mathbf{R}$ such that the set of points where f fails to be continuous is precisely F. (The concept of the interior of a set, discussed in Exercise 3.2.14, may be useful.)

(b) Now consider an open set O. Construct a function $g : \mathbf{R} \to \mathbf{R}$ whose set of discontinuous points is precisely O. (For this problem, the function in Exercise 4.3.12 may be useful.)

Solution.

(a) Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \cap F, \\ -1 & \text{if } x \in \mathbf{I} \cap F, \\ 0 & \text{if } x \notin F. \end{cases}$$

If $x \notin F$ then x belongs to the open set F^{c} and so there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \cap F^{\mathrm{c}}$. Notice that f vanishes on this proper open interval; it follows that f is continuous at x.

Suppose $x \in \mathbf{Q} \cap F$ and let $\delta > 0$ be given. We consider two cases.

Case 1. If $(x - \delta, x + \delta) \subseteq F$ then let y be an irrational in $(x - \delta, x + \delta)$ and observe that

$$f(y) = -1 \notin (0, 2) = (f(x) - 1, f(x) + 1).$$

Case 2. If $(x - \delta, x + \delta) \nsubseteq F$ then let $y \in (x - \delta, x + \delta)$ be such that $y \notin F$. It follows that

$$f(y) = 0 \notin (0, 2) = (f(x) - 1, f(x) + 1).$$

In either case we can find some $y \in V_\delta(x)$ such that $f(y) \notin V_1(f(x))$ and it follows that f is not continuous at x. A similar argument shows that f is not continuous at any $x \in \mathbf{I} \cap F$ either. We may conclude that the set of points where f fails to be continuous is precisely F.

(b) Let $d : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function and let $h : \mathbf{R} \to \mathbf{R}$ be the function given by

$$h(x) = \inf\{|x - a| : a \in O^{\mathrm{c}}\}.$$

In Exercise 4.3.12 we showed that h is continuous everywhere. Furthermore, since O^{c} is closed, Exercise 4.3.12 also shows that h satisfies $h(x) > 0$ for all $x \in O$ and $h(x) = 0$ for all $x \notin O$. Define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = d(x)h(x)$ and suppose that $x \in O$. Since

$h(x) > 0$ and h is continuous at x, Exercise 4.3.8 (c) shows that there is some $\delta > 0$ such that h is strictly positive on the interval $I = (x - \delta, x + \delta)$. It follows that for all $t \in I$ we have $d(t) = \frac{g(t)}{h(t)}$. If g were continuous at x then Theorem 4.3.4 would imply that d is continuous at x; since Dirichlet's function is nowhere-continuous, it must be the case that g is not continuous at x. Thus g is discontinuous on O.

Now suppose that $x \notin O$, so that $h(x) = 0$ and thus $g(x) = 0$. For any $y \in \mathbf{R}$ we then have

$$|g(y) - g(x)| = |g(y)| = |d(y)||h(y)| \leq |h(y)|.$$

Since h is continuous at x and $h(x) = 0$, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|y - x| < \delta \quad \Rightarrow \quad |h(y)| < \varepsilon.$$

It follows that $|g(y)| < \varepsilon$ for such y and thus g is continuous at x. We may conclude that the set of points where g fails to be continuous is precisely O.

4.4. Continuous Functions on Compact Sets

Exercise 4.4.1.

(a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbf{R}$.

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Solution.

(a) As Example 4.3.5 shows, any polynomial is continuous on all of $\mathbf{R}$.

(b) Define sequences (x_n) and (y_n) by $x_n = n + \frac{1}{n}$ and $y_n = n$ and observe that

$$|x_n - y_n| = \frac{1}{n} \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = 3n + \frac{3}{n} + \frac{1}{n^3} > 3.$$

Theorem 4.4.5 allows us to conclude that f is not uniformly continuous on $\mathbf{R}$.

(c) Suppose that $A \subseteq \mathbf{R}$ is a bounded subset of $\mathbf{R}$, so that there is some $M > 0$ such that $A \subseteq [-M, M]$. For any $x, y \in A$, it follows that

$$\left|x^2 + xy + y^2\right| \leq |x|^2 + |x||y| + \left|y^2\right| \leq 3M^2.$$

Let $\varepsilon > 0$ be given and let $\delta = \left(3M^2\right)^{-1}\varepsilon$. For any $x, y \in A$ such that $|x - y| < \delta$, we then have

$$\left|x^3 - y^3\right| = |x - y|\left|x^2 + xy + y^2\right| < 3M^2\delta = \varepsilon.$$

Thus f is uniformly continuous on A.

Exercise 4.4.2.

(a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?

(c) Is $h(x) = x \sin(1/x)$ uniformly continuous on $(0, 1)$?

Solution.

(a) Define sequences (x_n) and (y_n) by $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Observe that

$$|x_n - y_n| = \frac{1}{n} - \frac{1}{n+1} \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = 1.$$

Theorem 4.4.5 allows us to conclude that f is not uniformly continuous on $\mathbf{R}$.

(b) If a function is uniformly continuous on some $B \subseteq \mathbf{R}$ then it is also uniformly continuous on any subset $A \subseteq B$. The function $g(x) = \sqrt{x^2 + 1}$ is continuous on all of $\mathbf{R}$, hence uniformly continuous on the compact set $[0,1]$ (Theorem 4.4.7), and hence uniformly continuous on the subset $(0,1)$.

(c) Define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The continuity of h away from the origin is clear. As shown in Example 4.3.6, h is also continuous at the origin and thus continuous on all of $\mathbf{R}$. It follows that h is uniformly continuous on the compact set $[0,1]$ (Theorem 4.4.7) and hence uniformly continuous on the subset $(0,1)$.

Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1,\infty)$ but not on the set $(0,1]$.

Solution. For any $x, y \in [1,\infty)$ we have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{x+y}{x^2y^2}|x-y| = \left(\frac{1}{xy^2} + \frac{1}{x^2y}\right)|x-y| \leq 2|x-y|.$$

Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$. For any $x, y \in [1,\infty)$ such that $|x-y| < \delta$ we then have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq 2|x-y| < 2\delta = \varepsilon.$$

Thus f is uniformly continuous on $[1,\infty)$.

Define the sequences (x_n) and (y_n) in $(0,1]$ by $x_n = \frac{1}{\sqrt{n}}$ and $y_n = \frac{1}{\sqrt{n+1}}$. Observe that

$$|x_n - y_n| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = 1.$$

It follows from Theorem 4.4.5 that f is not uniformly continuous on $(0,1]$.

Exercise 4.4.4. Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a,b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a,b]$ (meaning $1/f$ has bounded range).

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

(c) If f is defined on $\mathbf{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbf{R}$.

Solution.

(a) This is true. Since f is continuous on the compact set $[a, b]$, Theorem 4.4.2 implies that there exist $x_0, x_1 \in [a, b]$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in [a, b]$. By assumption we have $f(x_0) > 0$ and thus

$$0 < f(x_0) \le f(x) \le f(x_1) \quad \Leftrightarrow \quad 0 < \frac{1}{f(x_1)} \le \frac{1}{f(x)} \le \frac{1}{f(x_0)}$$

for all $x \in [a, b]$, i.e. $1/f$ is bounded on $[a, b]$.

(b) This is true. Since A is bounded, there is a $K > 0$ such that $A \subseteq [-K, K]$, and since f is uniformly continuous on A, there is a $\delta > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < 1.$$

Let $N \in \mathbf{N}$ be such that $\frac{2K}{N} < \delta$ and for each $j \in \{1, 2, ..., N\}$ define

$$I_j = \left[-K + \frac{2K(j-1)}{N}, -K + \frac{2Kj}{N}\right],$$

so that $I_1 \cup \cdots \cup I_N = [-K, K]$. For $j \in \{1, 2, ..., N\}$, if $I_j \cap A \ne \emptyset$ then there exists some $a_j \in I_j \cap A$. Let

$$M = \max\{1 + |f(a_j)| : j \in \{1, 2, ..., N\} \text{ and } I_j \cap A \ne \emptyset\};$$

we are justified by Lemma L.3 in taking the maximum of this set as it is finite and must be non-empty, since if A is non-empty (which we may as well assume) there must be some j such that $I_j \cap A \ne \emptyset$.

Suppose $x \in A$. Since $I_1 \cup \cdots \cup I_N = [-K, K]$ and $A \subseteq [-K, K]$, there must be some $j \in \{1, 2, ..., N\}$ such that $x \in I_j \cap A$. Because $x, a_j \in I_j \cap A$ we then have

$$|x - a_j| \le |I_j| = \frac{2K}{N} < \delta \quad \Rightarrow \quad |f(x) - f(a_j)| < 1 \quad \Rightarrow \quad |f(x)| < 1 + |f(a_j)| \le M.$$

It follows that $f(A) \subseteq [-M, M]$, i.e. $f(A)$ is bounded.

(c) This is false. Let $f : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function, i.e.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

For any subset $A \subseteq \mathbf{R}$, the only possibilities for $f(A)$ are $\emptyset, \{0\}, \{1\}$, and $\{0, 1\}$; each of these is compact. However, f is nowhere-continuous.

Exercise 4.4.5. Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Solution. Let $\varepsilon > 0$ be given. There exist positive real numbers δ_1 and δ_2 such that

$$x, y \in (a, b] \text{ and } |x - y| < \delta_1 \quad \Rightarrow \quad |g(x) - g(y)| < \tfrac{\varepsilon}{2},$$
$$x, y \in [b, c) \text{ and } |x - y| < \delta_2 \quad \Rightarrow \quad |g(x) - g(y)| < \tfrac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $x, y \in (a, c)$ are such that $|x - y| < \delta$. There are four cases.

Case 1. If $x, y \in (a, b]$, then since $|x - y| < \delta \leq \delta_1$ we have $|g(x) - g(y)| < \frac{\varepsilon}{2} < \varepsilon$.

Case 2. If $x, y \in [b, c)$, then since $|x - y| < \delta \leq \delta_2$ we have $|g(x) - g(y)| < \frac{\varepsilon}{2} < \varepsilon$.

Case 3. If $x \in (a, b]$ and $y \in [b, c)$, then note that

$$|x - b| \leq |x - y| < \delta \leq \delta_1 \quad \Rightarrow \quad |g(x) - g(b)| < \tfrac{\varepsilon}{2},$$
$$|b - y| \leq |x - y| < \delta \leq \delta_2 \quad \Rightarrow \quad |g(b) - g(y)| < \tfrac{\varepsilon}{2}.$$

It follows that $|g(x) - g(y)| \leq |g(x) - g(b)| + |g(b) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Case 4. The case where $x \in [b, c)$ and $y \in (a, b]$ is handled similarly to Case 3.

In any case we have $|g(x) - g(y)| < \varepsilon$. Thus g is uniformly continuous on (a, c).

Exercise 4.4.6. Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(b) A uniformly continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(c) A continuous function $f : [0, \infty) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution.

(a) Let $f : (0, 1) \to \mathbf{R}$ be given by $f(x) = \frac{1}{x}$ and consider the Cauchy sequence (x_n) given by $x_n = \frac{1}{n+1}$. Notice that $f(x_n) = n + 1$, which is not convergent and hence not Cauchy.

(b) This is impossible, as we will show in Exercise 4.4.13 (a).

(c) This is impossible. Let $f : [0, \infty) \to \mathbf{R}$ be continuous and let (x_n) be a Cauchy sequence contained in $[0, \infty)$; by Theorem 2.6.4 we must have $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$. Since $[0, \infty)$ is a closed set we must have $x \in [0, \infty)$, and because f is continuous at x it follows that $\lim_{n\to\infty} f(x_n) = f(x)$. Thus $(f(x_n))$ is a Cauchy sequence (Theorem 2.6.4).

Exercise 4.4.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution. Note that f is continuous on the compact set $[0,1]$ and hence is uniformly continuous on $[0,1]$ (Theorem 4.4.7). Note further that for any $x, y \in [1, \infty)$ we have

$$\left|\sqrt{x} - \sqrt{y}\right| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}|x-y|.$$

It is now straightforward to show that f is uniformly continuous on $[1, \infty)$ (see, for example, Exercise 4.4.9). By an argument analogous to the one given in Exercise 4.4.5, we may now conclude that f is uniformly continuous on $[0, \infty)$.

Exercise 4.4.8. Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0,1]$ with range $(0,1)$.

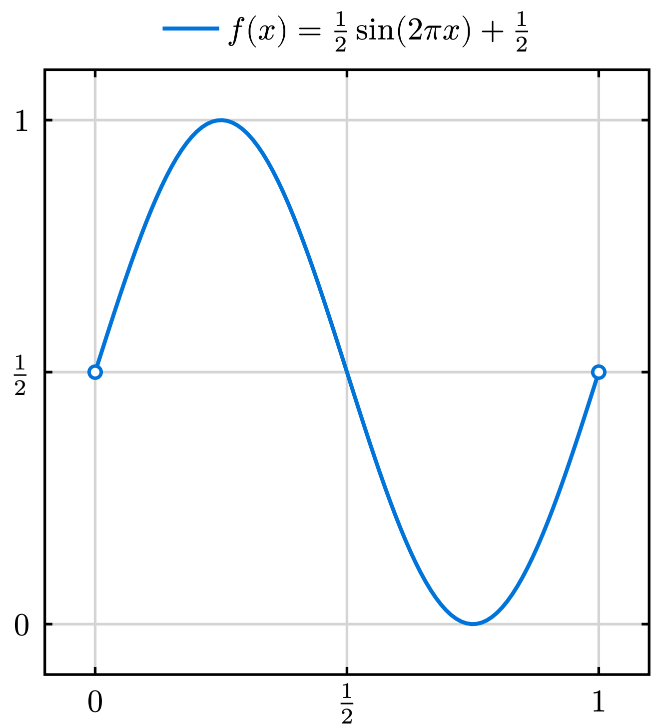
(b) A continuous function defined on $(0,1)$ with range $[0,1]$.

(c) A continuous function defined on $(0,1]$ with range $(0,1)$.

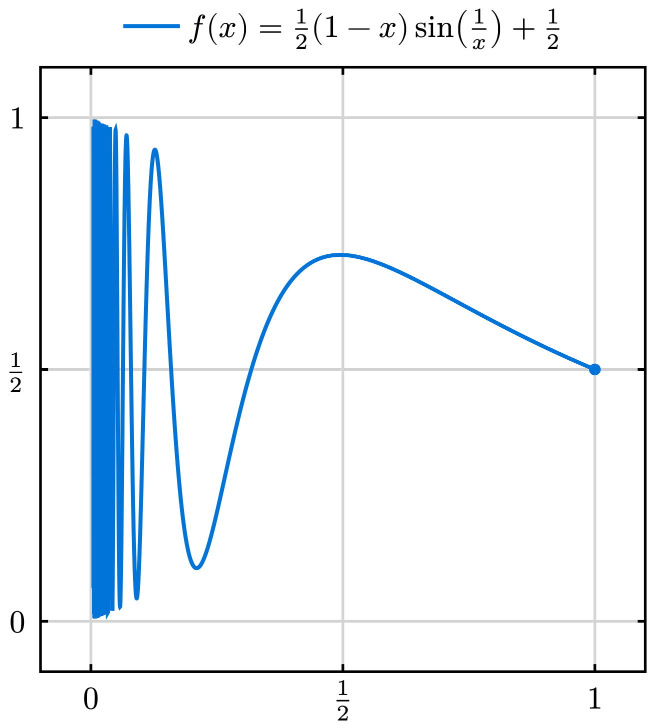
Solution.

(a) This is impossible. If $f : [0,1] \to \mathbf{R}$ is continuous then since $[0,1]$ is compact, the image of f must be compact (Theorem 4.4.1). However, $(0,1)$ is not compact.

(b) Consider $f : (0,1) \to \mathbf{R}$ given by $f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{2}$, which has range equal to $[0,1]$.



(c) Consider $f : (0,1] \to \mathbf{R}$ given by $f(x) = \frac{1}{2}(1-x)\sin\left(\frac{1}{x}\right) + \frac{1}{2}$, which has range equal to $(0,1)$.



Exercise 4.4.9 (Lipschitz Functions). A function $f : A \to \mathbf{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution.

(a) Since f is Lipschitz, there is an $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$. Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{M}$. For any $x, y \in A$ satisfying $|x - y| < \delta$, we then have

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \varepsilon.$$

Thus f is uniformly continuous on A.

(b) The converse statement is not true. Consider $f : [0, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$. As we showed in Exercise 4.4.7, this function is uniformly continuous on $[0, \infty)$. However, we claim that f is not Lipschitz on $[0, \infty)$. To show this, for each $M > 0$ we need to find some $x \neq y \in [0, \infty)$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| > M.$$

Given $M > 0$, let $x = \frac{1}{4M^2}$ and $y = 0$ and observe that

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\frac{1}{2M}}{\frac{1}{4M^2}} \right| = 2M > M.$$

Thus f is not Lipschitz on $[0, \infty)$.

Exercise 4.4.10. Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)}, \quad f(g(x))\ ?$$

(Assume that the quotient and the composition are properly defined and thus at least continuous.)

Solution. We claim that $f + g$ is uniformly continuous on A. To see this, let $\varepsilon > 0$ be given. There exist $\delta_1, \delta_2 > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_1 \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2},$$
$$x, y \in A \text{ and } |x - y| < \delta_2 \quad \Rightarrow \quad |g(x) - g(y)| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and observe that for any $x, y \in A$ satisfying $|x - y| < \delta$ we have

$$|f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f + g$ is uniformly continuous on A.

The product fg need not be uniformly continuous. For a counterexample, consider $f, g : \mathbf{R} \to \mathbf{R}$ given by $f(x) = g(x) = x$. These functions are clearly Lipschitz and hence uniformly continuous on all of $\mathbf{R}$ (Exercise 4.4.9). However, the product $f(x)g(x) = x^2$ is not uniformly continuous on $\mathbf{R}$ this can be seen using the same sequences as in Exercise 4.4.1 (b) and appealing to Theorem 4.4.5.

The quotient f/g need not be uniformly continuous. For a counterexample, consider $f, g : (0, 1] \to \mathbf{R}$ given by $f(x) = 1$ and $g(x) = x$. Both are uniformly continuous, but the quotient $f(x)/g(x) = 1/x$ is not (Exercise 4.4.2 (a)).

Suppose that $g(A) \subseteq A$, so that the composition $f \circ g : A \to \mathbf{R}$ is well-defined. We claim that this composition is uniformly continuous. To see this, let $\varepsilon > 0$ be given. There exists a $\delta_2 > 0$ such that

$$s, t \in A \text{ and } |s - t| < \delta_2 \quad \Rightarrow \quad |f(s) - f(t)| < \varepsilon.$$

There then exists a $\delta_1 > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_1 \quad \Rightarrow \quad |g(x) - g(y)| < \delta_2.$$

By assumption, $x, y \in A$ implies $g(x), g(y) \in A$. Thus

$$\begin{aligned} x, y \in A \text{ and } |x - y| < \delta_1 \quad &\Rightarrow \quad g(x), g(y) \in A \text{ and } |g(x) - g(y)| < \delta_2 \\ &\Rightarrow \quad |f(g(x)) - f(g(y))| < \varepsilon. \end{aligned}$$

It follows that $f \circ g$ is uniformly continuous on A.

Exercise 4.4.11 (Topological Characterization of Continuity). Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set.

Solution. Suppose g is continuous and $O \subseteq \mathbf{R}$ is an open set. Fix $c \in g^{-1}(O)$, so that $g(c) \in O$. Since O is open there exists an $\varepsilon > 0$ such that $V_\varepsilon(g(c)) \subseteq O$, and since g is continuous at c there is a $\delta > 0$ such that $x \in V_\delta(c)$ implies $g(x) \in V_\varepsilon(g(c)) \subseteq O$ (Theorem 4.3.2 (ii)). In other words, any $x \in V_\delta(c)$ also belongs to $g^{-1}(O)$, so that $V_\delta(c) \subseteq g^{-1}(O)$. It follows that $g^{-1}(O)$ is an open set.

Now suppose that $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set. Fix $c \in \mathbf{R}$ and let $\varepsilon > 0$ be given. The set $V_\varepsilon(g(c))$ is open, so by assumption the set $g^{-1}[V_\varepsilon(g(c))]$ is also open. Certainly we have $c \in g^{-1}[V_\varepsilon(g(c))]$, so there exists a $\delta > 0$ such that $V_\delta(c) \subseteq g^{-1}[V_\varepsilon(g(c))]$. It follows that if $x \in V_\delta(c)$ then $g(x) \in V_\varepsilon(g(c))$; Theorem 4.3.2 (ii) allows us to conclude that g is continuous at each $c \in \mathbf{R}$.

Exercise 4.4.12. Review Exercise 4.4.11, and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:

(a) $f^{-1}(B)$ is finite whenever B is finite.

(b) $f^{-1}(K)$ is compact whenever K is compact.

(c) $f^{-1}(A)$ is bounded whenever A is bounded.

(d) $f^{-1}(F)$ is closed whenever F is closed.

Solution.

(a) This is false. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 0$, which satisfies $f^{-1}(\{0\}) = \mathbf{R}$.

(b) This is false; see part (a) for a counterexample.

(c) This is false; see part (a) for a counterexample.

(d) This is true. If F is closed then F^c is open. Since f is continuous, it follows from Exercise 4.4.11 that $f^{-1}(F^c)$ is open and thus $\left(f^{-1}(F^c)\right)^c$ is closed. This set is nothing but $f^{-1}(F)$:

$$x \in \left(f^{-1}(F^c)\right)^c \quad \Leftrightarrow \quad x \notin f^{-1}(F^c) \quad \Leftrightarrow \quad f(x) \notin F^c \quad \Leftrightarrow \quad f(x) \in F \quad \Leftrightarrow \quad x \in f^{-1}(F).$$

Exercise 4.4.13 (Continuous Extension Theorem).

(a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbf{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show that $f(x_n)$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Solution.

(a) Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there is a $\delta > 0$ such that for any $x, y \in A$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Since $(x_n) \subseteq A$ is a Cauchy sequence, there is an $N \in \mathbf{N}$ such that for all $n > m \geq N$ we have $|x_n - x_m| < \delta$, which implies that $|f(x_n) - f(x_m)| < \varepsilon$. Thus $(f(x_n))$ is also a Cauchy sequence.

(b) Suppose that g is uniformly continuous on (a, b). Define a sequence $a_n = a + \frac{b-a}{2n}$, so that (a_n) is contained in (a, b) and satisfies $\lim_{n\to\infty} a_n = a$. Because (a_n) is Cauchy, part (a) implies that the sequence $(g(a_n))$ is also Cauchy and hence convergent, say $\lim_{n\to\infty} g(a_n) = y \in \mathbf{R}$. Define $g(a) = y$.

We claim that this extended g is continuous at a. Let (x_n) be a sequence contained in (a, b) such that $\lim_{n\to\infty} x_n = a$ and let $\varepsilon > 0$ be given. Since g is uniformly continuous on (a, b), there is a $\delta > 0$ such that for any $x, y \in (a, b)$ satisfying $|x - y| < \delta$ we have $|g(x) - g(y)| < \varepsilon$. Note that $\lim_{n\to\infty}|x_n - a_n| = 0$ since both (x_n) and (a_n) converge to a. It follows that there exists an $N \in \mathbf{N}$ such that $|x_n - a_n| < \delta$ whenever $n \geq N$, which implies $|g(x_n) - g(a_n)| < \varepsilon$. Thus $\lim_{n\to\infty}|g(x_n) - g(a_n)| = 0$. Combining this with $\lim_{n\to\infty} g(a_n) = g(a)$, we see that $\lim_{n\to\infty} g(x_n) = g(a)$ also. Hence g is continuous at a.

An analogous argument shows that we can also continuously extend g to be defined at b by considering the sequence $b_n = b - \frac{b-a}{2n}$.

For the converse implication, we apply Theorem 4.4.7 to see that g is uniformly continuous on the compact set $[a, b]$ and hence uniformly continuous on the subset (a, b).

Exercise 4.4.14. Construct an alternate proof of Theorem 4.4.7 using the open cover characterization of compactness from the Heine-Borel Theorem (Theorem 3.3.8 (iii)).

Solution. Suppose $f : K \to \mathbf{R}$ is continuous, where K is compact. Let $\varepsilon > 0$ be given. Since f is continuous on K, for each $t \in K$ there exists a $\delta_t > 0$ such that

$$x \in K \text{ and } |x - t| < \delta_t \quad \Rightarrow \quad |f(x) - f(t)| < \frac{\varepsilon}{2}.$$

Observe that the collection $\{V_{\delta_t/2}(t) : t \in K\}$ forms an open cover of K. Because K is compact there must exist a finite subcover $\{V_{\delta_{t_1}/2}(t_1), ..., V_{\delta_{tn}/2}(t_n)\}$. Let $\delta = \min\{\delta_{t_1}, ..., \delta_{t_n}\}$ and suppose that $x, y \in K$ are such that $|x - y| < \frac{\delta}{2}$. There is a $j \in \{1, ..., n\}$ such that $x \in V_{\delta_{t_j}/2}(t_j)$, so that $|x - t_j| < \delta_{t_j}/2 < \delta_{t_j}$ and thus $|f(x) - f(t_j)| < \frac{\varepsilon}{2}$. Note that

$$|y - t_j| \leq |x - y| + |x - t_j| < \frac{\delta}{2} + \frac{\delta_{t_j}}{2} \leq \frac{\delta_{t_j}}{2} + \frac{\delta_{t_j}}{2} = \delta_{t_j}.$$

It follows that $|f(y) - f(t_j)| < \frac{\varepsilon}{2}$ and hence that

$$|f(x) - f(y)| \leq |f(x) - f(t_j)| + |f(y) - f(t_j)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus f is uniformly continuous on K.

4.5. The Intermediate Value Theorem

Exercise 4.5.1. Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2.

Solution. Let $f : [a, b] \to \mathbf{R}$ be continuous and let $L \in \mathbf{R}$ be such that either $f(a) < L < f(b)$ or $f(b) < L < f(a)$; our aim is to show that there exists $c \in (a, b)$ such that $f(c) = L$. Theorem 3.4.7 shows that $[a, b]$ is connected and hence, by Theorem 4.5.2, the image $f([a, b])$ is also connected. Certainly $f(a), f(b) \in f([a, b])$, so Theorem 3.4.7 implies that $L \in f([a, b])$, i.e. there exists $c \in [a, b]$ such that $f(c) = L$. In fact, because $f(a) \neq L$ and $f(b) \neq L$, we have $c \in (a, b)$.

Exercise 4.5.2. Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

(b) A continuous function defined on a closed interval with range equal to an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbf{R}$.

(d) A continuous function defined on all of $\mathbf{R}$ with range equal to $\mathbf{Q}$.

Solution. (I am not sure if Abbott allows unbounded intervals here.)

(a) If we allow unbounded intervals then $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$ is an example of such a function. For bounded intervals, see Exercise 4.4.8 (b) for an example of such a function.

(b) If we allow unbounded intervals then $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$ is an example of such a function. If we do not allow unbounded intervals then such a function cannot exist by Theorem 4.4.1 (Preservation of Compact Sets), since a bounded open interval is not closed.

(c) If we allow unbounded intervals, then $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = \max\{0, x\}$ is an example of such a function; the image of f is $[0, \infty)$. For bounded intervals, consider the function $f : (0, 2) \to \mathbf{R}$ given by $f(x) = \frac{1}{x(2-x)}$; the image of f is $[1, \infty)$.

(d) This is impossible. $\mathbf{R}$ is connected (Theorem 3.4.7) and so its image under a continuous function must also be connected (Theorem 4.5.2), but $\mathbf{Q}$ is not connected (Theorem 3.4.7).

Exercise 4.5.3. A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Solution. First, let us prove the following lemma.

Lemma L.12. Suppose $a < b$ and $f : [a, b] \to \mathbf{R}$ is increasing.

(i) If $c \in (a, b]$ then
$$\lim_{x \to c^-} f(x) = \sup\{f(x) : a < x < c\}.$$
(ii) If $c \in [a, b)$ then
$$\lim_{x \to c^+} f(x) = \inf\{f(x) : c < x < b\}.$$

Proof. We will prove (i); the proof of (ii) is similar. Note that since f is increasing, we have $f([a, b]) \subseteq [f(a), f(b)]$; it follows that $\{f(x) : a < x < c\}$ is bounded and non-empty, so $s := \sup\{f(x) : a < x < c\}$ exists. Let $\varepsilon > 0$ be given. By Lemma 1.3.8 there exists a $y \in (a, c)$ such that $s - \varepsilon < f(y) \leq s$. Because f is increasing, it follows that
$$x \in (y, c) \quad \Rightarrow \quad s - \varepsilon < f(y) \leq f(x) \leq s.$$
In other words, letting $\delta = c - y$, for any x satisfying $c - \delta < x < c$ it follows that $|f(x) - s| < \varepsilon$. Thus $\lim_{x \to c^-} f(x) = s$. □

Returning to the exercise, we will now prove the contrapositive statement: if f is increasing and not continuous on $[a, b]$, then f does not satisfy the intermediate value property. Suppose therefore that f is not continuous at some $c \in [a, b]$, i.e. suppose that $\lim_{x \to c} f(x) \neq f(c)$ (Theorem 4.3.2 (iv)).

Case 1. Suppose $c \in (a, b)$. Since f is increasing on $[a, b]$, Lemma L.12 implies that both of the one-sided limits exist:
$$\alpha := \lim_{x \to c^-} f(x) = \sup\{f(x) : a < x < c\},$$
$$\beta := \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x < b\}.$$
By Exercise 4.2.10 (b), it must be the case that at least one of these limits is not equal to $f(c)$. Because f is increasing, we must then have $\alpha < \beta$; it follows that the infinite set $(\alpha, \beta) \setminus \{f(c)\}$, which is contained in $[f(a), f(b)]$, does not intersect the image of f. Thus f does not satisfy the intermediate value property on $[a, b]$.

Case 2. Suppose $c = a$, i.e. f is not continuous at a. Since f is increasing on $[a, b]$, Lemma L.12 implies that the limit from the right exists:
$$\beta := \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x < b\}.$$

Because a is the minimum element of the domain of f we have

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \beta,$$

and since f is not continuous at a and increasing on $[a,b]$ it must then be the case that $f(a) < \beta$. It follows that the infinite set $(f(a),\beta)$, which is contained in $[f(a), f(b)]$, does not intersect the image of f. Thus f does not satisfy the intermediate value property on $[a,b]$.

Case 3. If f fails to be continuous at b, then an argument similar to the one given in Case 2, this time using the limit $x \to b^-$, shows that f does not satisfy the intermediate value property on $[a,b]$.

Exercise 4.5.4. Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x \in A : f(x) = f(y) \text{ for some } y \neq x \text{ and } y \in A\}.$$

Show F is either empty or uncountable.

Solution. It will suffice to show that if F is not empty then F is uncountable. Suppose therefore that there exist $x, y \in A$ such that $x < y$ and $g(x) = g(y)$. If g is constant on $[x,y]$ then F contains the uncountable subset $[x,y]$ and so must itself be uncountable. Otherwise, there exists some $a \in (x,y)$ such that $g(a) \neq g(x)$. Define an open interval

$$I = (\min\{g(x), g(a)\}, \max\{g(x), g(a)\})$$

and note that I is non-empty since $g(a) \neq g(x)$. Since g is continuous on A, the Intermediate Value Theorem (Theorem 4.5.1) implies that for each $t \in I$ there exist $x_t \in (x,a)$ and $y_t \in (a,y)$ such that $g(x_t) = g(y_t) = t$, so that $x_t \in F$. Because g is a function, each $t \in I$ corresponds to a distinct $x_t \in F$, i.e. the map $I \to F$ given by $t \mapsto x_t$ is injective. Since I is uncountable, it then follows that F is uncountable.

Exercise 4.5.5.

(a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Solution.

(a) (Here is the start of the proof from the textbook.) To simplify matters a bit, let's consider the special case where f is a continuous function satisfying $f(a) < 0 < f(b)$ and show that $f(c) = 0$ for some $c \in (a,b)$. First let

$$K = \{x \in [a,b] : f(x) \leq 0\}.$$

Notice that K is bounded above by b, and $a \in K$ so K is not empty. Thus we may appeal to the Axiom of Completeness to assert that $c = \sup K$ exists.

There are three cases to consider:

$$f(c) > 0, \quad f(c) < 0, \quad \text{and} \quad f(c) = 0.$$

Case 1. Suppose that $f(c) > 0$. Since f is continuous at c, there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$ (see Exercise 4.3.8 (c)). This implies the existence of a $t \in (c - \delta, c) \cap [a, b]$ such that t is an upper bound of K, which contradicts that c is the supremum of K.

Case 2. Suppose that $f(c) < 0$. Since f is continuous at c, there is a $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$ (see Exercise 4.3.8 (c)). This implies the existence of a $t \in (c, c + \delta) \cap [a, b]$ such that t belongs to K, which contradicts that c is the supremum of K.

So the only possibility is that $f(c) = 0$; note that c lies strictly between a and b since $f(a) < 0 < f(b)$.

The general case of the Intermediate Value Theorem can be obtained from this special case by considering either the function $g(x) = f(x) - L$ if $f(a) < f(b)$ or the function $g(x) = L - f(x)$ if $f(a) > f(b)$.

(b) (Here is the start of the proof from the textbook.) Again, consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint

$$z = (a + b)/2.$$

If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and nonnegative at the right.

We repeat this procedure inductively, obtaining a sequence $(I_n = [a_n, b_n])_{n=1}^{\infty}$ of nested intervals such that $f(a_n) < 0, f(b_n) \geq 0$, and $|I_n| = 2^{-n}(b - a)$ for all $n \in \mathbf{N}$. We can now appeal to the Nested Interval Property (Theorem 1.4.1) to assert that $\bigcap_{n=1}^{\infty} I_n = \{c\}$ for some $c \in [a, b]$ (the intersection is non-empty as the intervals are closed and nested, and the intersection is a singleton since $\lim_{n\to\infty} |I_n| = 0$); furthermore, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c.$$

Since f is continuous at c, it follows that

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(c).$$

The Order Limit Theorem (Theorem 2.3.4) implies that $f(c) \leq 0$, since $f(a_n) < 0$ for all $n \in \mathbf{N}$, and that $f(c) \geq 0$, since $f(b_n) \geq 0$ for all $n \in \mathbf{N}$. Thus $f(c) = 0$.

Again, c lies strictly between a and b since $f(a) < 0 < f(b)$, and the general case of the Intermediate Value Theorem can be obtained from this special case by considering either the function $g(x) = f(x) - L$ if $f(a) < f(b)$ or the function $g(x) = L - f(x)$ if $f(a) > f(b)$.

Exercise 4.5.6. Let $f : [0, 1] \to \mathbf{R}$ be continuous with $f(0) = f(1)$.

(a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

(b) Show that for each $n \in \mathbf{N}$ there exist $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(c) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

Solution.

(a) Define $g : \left[0, \frac{1}{2}\right] \to \mathbf{R}$ by $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$ and note that g is continuous by Theorems 4.3.4 and 4.3.9. If $g(0) = 0$ then $f(0) = f\left(\frac{1}{2}\right)$ and we are done. Otherwise, note that

$$g(0) = f(0) - f\left(\tfrac{1}{2}\right) = f(1) - f\left(\tfrac{1}{2}\right) = -\left(f\left(\tfrac{1}{2}\right) - f(1)\right) = -g\left(\tfrac{1}{2}\right).$$

It follows that $g(0)$ and $g\left(\frac{1}{2}\right)$ have opposite signs. The Intermediate Value Theorem (Theorem 4.5.1) now implies that there exists a $c \in \left(0, \frac{1}{2}\right)$ such that $g(c) = 0$, i.e. $f(c) = f\left(c + \frac{1}{2}\right)$.

(b) For $n = 1$, we can take $x_1 = 0$ and $y_1 = 1$. For $n \geq 2$, define $g : \left[0, \frac{n-1}{n}\right] \to \mathbf{R}$ by $g(x) = f(x) - f\left(x + \frac{1}{n}\right)$ and note that g is continuous by Theorems 4.3.4 and 4.3.9. If $g(0) = 0$ then $f(0) = f\left(\frac{1}{n}\right)$ and we are done. Otherwise, note that

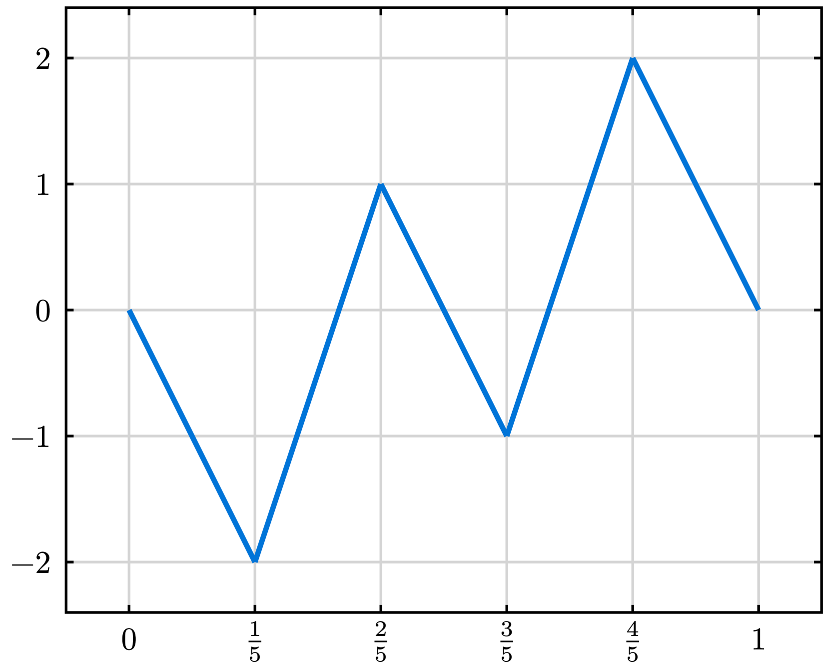
$$\begin{aligned} g(0) &= f(0) - f\left(\tfrac{1}{n}\right), \\ g\left(\tfrac{1}{n}\right) &= f\left(\tfrac{1}{n}\right) - f\left(\tfrac{2}{n}\right), \\ g\left(\tfrac{2}{n}\right) &= f\left(\tfrac{2}{n}\right) - f\left(\tfrac{3}{n}\right), \\ &\vdots \\ g\left(\tfrac{n-1}{n}\right) &= f\left(\tfrac{n-1}{n}\right) - f(1). \end{aligned}$$

Since $f(0) = f(1)$, this implies that

$$g(0) + g\left(\tfrac{1}{n}\right) + g\left(\tfrac{2}{n}\right) + \cdots + g\left(\tfrac{n-1}{n}\right) = 0.$$

Because $g(0) \neq 0$, there must exist some $k \in \{1, ..., n-1\}$ such that $g\left(\frac{k}{n}\right)$ has the opposite sign to $g(0)$. The Intermediate Value Theorem (Theorem 4.5.1) now implies that there exists a $c \in \left(0, \frac{k}{n}\right)$ such that $g(c) = 0$, i.e. $f(c) = f\left(c + \frac{1}{n}\right)$. Thus we can take $x_n = c$ and $y_n = c + \frac{1}{n}$.

(c) Consider the following piecewise linear function $f : [0, 1] \to \mathbf{R}$.



This function has the property that $f\left(x+\frac{2}{5}\right)-f(x)=1$ for every $x \in \left[0, \frac{3}{5}\right]$, so that there cannot possibly exist $x, y \in [0,1]$ satisfying $|x-y|=\frac{2}{5}$ and $f(x)=f(y)$.

Exercise 4.5.7. Let f be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that f must have a fixed point; that is, show $f(x)=x$ for at least one value of $x \in [0,1]$.

Solution. Define $g : [0,1] \to \mathbf{R}$ by $g(x)=f(x)-x$ and note that g is continuous by Theorem 4.3.4. Furthermore, fixed points of f correspond precisely to zeros of g. If $g(0)=0$ or $g(1)=0$, then we are done. Suppose therefore that $g(0) \neq 0$ and $g(1) \neq 0$. Since $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, it must then be the case that $0 < f(0) \leq 1$ and $0 \leq f(1) < 1$, which implies that $g(0)$ is positive and $g(1)$ is negative. The Intermediate Value Theorem (Theorem 4.5.1) can now be applied to obtain some $x \in (0,1)$ such that $g(x)=0$.

Exercise 4.5.8 (Inverse functions). If a function $f : A \to \mathbf{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y)=x$ where $y=f(x)$.

Show that if f is continuous on an interval $[a,b]$ and one-to-one, then f^{-1} is also continuous.

Solution. Here is a useful corollary of Exercise 4.4.11.

Lemma L.13. Suppose $h : A \to \mathbf{R}$ has the property that $h^{-1}(B)$ is closed for every closed set $B \subseteq \mathbf{R}$. Then h is continuous.

Proof. Let $U \subseteq \mathbf{R}$ be open, so that U^{c} is closed. By assumption $h^{-1}(U^{c})$ is closed;

notice that

$$h^{-1}(U^{c}) = \left(h^{-1}(U)\right)^{c}.$$

Thus $h^{-1}(U)$ is open. It follows from Exercise 4.4.11 that h is continuous. $\square$

Suppose $f : [a, b] \to E \subseteq \mathbf{R}$ is continuous and bijective, and let $h : E \to [a, b]$ be the inverse of f. If $B \subseteq \mathbf{R}$ is closed then $B \cap [a, b]$ is closed and bounded, hence compact (Heine-Borel Theorem). It follows that $f(B \cap [a, b])$ is compact and hence closed. Using that f and h are mutual inverses, notice that

$$f(B \cap [a, b]) = f(B) \cap f([a, b]) = h^{-1}(B) \cap E = h^{-1}(B).$$

Thus $h^{-1}(B)$ is closed whenever $B \subseteq \mathbf{R}$ is closed; it follows from Lemma L.13 that h is continuous.

4.6. Sets of Discontinuity

Exercise 4.6.1. Using modifications of these functions, construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

(b) $D_f = \{x : 0 < x \leq 1\}$.

Solution.

(a) Since $\mathbf{Z}^c$ is an open set, the construction given in Exercise 4.3.14 (b) will result in an f such that $D_f = \mathbf{Z}^c$.

(b) By Exercise 4.3.14, there exist functions $g, h : \mathbf{R} \to \mathbf{R}$ such that $D_g = \left(0, \frac{1}{2}\right)$ and $D_h = \left[\frac{1}{2}, 1\right]$. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = g(x) + h(x)$; it follows from Theorem 4.3.4 that $D_f = (0, 1]$.

Exercise 4.6.2. Given a countable set $A = \{a_1, a_2, a_3, ...\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

Solution. Our claim is that $D_f = A$. First, fix $c \notin A$; we will show that f is continuous at c. Let $\varepsilon > 0$ be given and let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \varepsilon$. Consider the set

$$E = \{|c - a_n| : 1 \leq n \leq N\}.$$

This set is non-empty and finite and thus has a minimum (by Lemma L.3), say $\delta = \min E$. Each element of E must be strictly positive as $c \notin A$ and hence δ is also strictly positive. Furthermore, the interval $(c - \delta, c + \delta)$ has the property that if $a_n \in (c - \delta, c + \delta)$ then $n > N$ (otherwise δ would not be the minimum of E) and thus

$$|f(a_n) - f(c)| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

If $x \in (c - \delta, c + \delta)$ and $x \notin A$ then $|f(x) - f(c)| = 0 < \varepsilon$. We have now shown that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x \in (c - \delta, x + \delta) \quad \Rightarrow \quad f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon).$$

Thus f is continuous at each $c \notin A$.

Now fix $a_n \in A$. We will show that f is not continuous at a_n. Let $\varepsilon = \frac{1}{n} > 0$ and let $\delta > 0$ be given. Because the interval $(a_n - \delta, a_n + \delta)$ is uncountable and A is countable, it must be the case that there exists an $x \in (a_n - \delta, a_n + \delta)$ such that $x \notin A$. It follows that

$$|f(x) - f(a_n)| = \frac{1}{n} = \varepsilon.$$

Thus f is not continuous at a_n. We may conclude that $D_f = A$, as claimed.

Exercise 4.6.3. State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L.$$

Solution. See Exercise 4.2.10 (a).

Exercise 4.6.4. Supply a proof for this proposition.

Solution. See Exercise 4.2.10 (b).

Exercise 4.6.5. Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Solution. Suppose $f : \mathbf{R} \to \mathbf{R}$ is monotone. (For simplicity, we will assume that the domain of f is all of $\mathbf{R}$. A more general statement can certainly be made for monotone functions $A \to \mathbf{R}$ defined on any domain $A \subseteq \mathbf{R}$, but Abbott's definitions of left- and right-hand limits are slightly awkward here. For example, if $f : [0,1] \to \mathbf{R}$ is a function, then Abbott's definition of the left-hand limit of f at 0 implies that $\lim_{x \to 0^-} f(x) = L$ for *any* $L \in \mathbf{R}$: we may choose any $\delta > 0$ we like and obtain a statement beginning with $(\forall x \in \emptyset)$, which is always true. It would be better not to talk about $\lim_{x \to 0^-} f(x)$ at all in such a case.)

First, note that a small modification of Lemma L.12 shows that if f is increasing then for each $c \in \mathbf{R}$,

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \quad \text{and} \quad \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x\}.$$

(If f is decreasing then the supremum and the infimum should be swapped.) So for a monotone function $f : \mathbf{R} \to \mathbf{R}$, the left- and right-hand limits at some point $c \in \mathbf{R}$ always exist. It follows that if f is discontinuous at c then it must be the case that these left- and right-hand limits are not equal (Theorem 4.6.3/Exercise 4.6.4), i.e. f has a jump discontinuity at c.

Exercise 4.6.6. Construct a bijection between the set of jump discontinuities of a monotone function f and a subset of $\mathbf{Q}$. Conclude that D_f for a monotone function f must either be finite or countable, but not uncountable.

Solution. Suppose $f : \mathbf{R} \to \mathbf{R}$ is monotone increasing (if f is decreasing then consider $-f$) and let D_f be the set of jump discontinuities of f (by Exercise 4.6.5, D_f is the set of all discontinuities of f). Fix $c \in D_f$. As we showed in Exercise 4.6.5, we have

$$\ell_c := \lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \quad \text{and} \quad u_c := \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x\}.$$

Since f is discontinuous at c and increasing we must have $\ell_c < u_c$ and thus (ℓ_c, u_c) is a proper open interval. If $d \in D_f$ is such that $c < d$ then $u_c \leq f\left(\frac{c+d}{2}\right) \leq \ell_d$, so that the open intervals (ℓ_c, u_c) and (ℓ_d, u_d) are disjoint. It follows that the set

$$\{(\ell_c, u_c) : c \in D_f\}$$

consists of pairwise disjoint open intervals. Given this, for each $c \in D_f$ we can choose a rational number $r_c \in (\ell_c, u_c)$ and be sure that the function $g : D_f \to \mathbf{Q}$ given by $c \mapsto r_c$ is injective. This sets up a bijection between D_f and $g(D_f) \subseteq \mathbf{Q}$. It follows from Theorems 1.5.6 and 1.5.7 (i) that D_f is finite or countable, but not uncountable.

Exercise 4.6.7.

(a) Show that in each of the above cases we get an F_σ set as the set where the function is discontinuous.

(b) Show that the two sets of discontinuity in Exercise 4.6.1 are F_σ sets.

Solution.

(a) For Dirichlet's function, $\mathbf{R}$ is a closed set. For the modified Dirichlet function, we have

$$\mathbf{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left(-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \infty\right).$$

For Thomae's function we have

$$\mathbf{Q} = \bigcup_{q \in \mathbf{Q}} \{q\}.$$

(b) Observe that

$$\mathbf{Z}^c = \bigcup_{(m,n) \in \mathbf{Z} \times \mathbf{N}} \left[m + \frac{1}{n+1}, m + 1 - \frac{1}{n+1}\right] \quad \text{and} \quad (0, 1] = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right].$$

Exercise 4.6.8. Prove that, for a fixed $\alpha > 0$, the set D_f^α is closed.

Solution. First, let us write down the negation of α-continuity. A function f is not α-continuous at a point $x \in \mathbf{R}$ if for all $\delta > 0$ there exist $y, z \in (x - \delta, x + \delta)$ such that $|f(y) - f(z)| \geq \alpha$.

To show that D_f^α is closed, let (x_n) be a sequence contained in D_f^α such that $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$. Our aim is to show that f is not α-continuous at x. Let $\delta > 0$ be given. Since $\lim_{n\to\infty} x_n = x$ there is an $N \in \mathbf{N}$ such that $x_N \in \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$, and since f is not α -continuous at x_N there exist $y, z \in \left(x_N - \frac{\delta}{2}, x_N + \frac{\delta}{2}\right)$ such that $|f(y) - f(z)| \geq \alpha$. The triangle inequality shows that $y, z \in (x - \delta, x + \delta)$ and thus f is not α-continuous at x, i.e. $x \in D_f^\alpha$.

It follows that D_f^α contains its limit points and hence that D_f^α is a closed set.

Exercise 4.6.9. If $\alpha < \alpha'$, show that $D_f^{\alpha'} \subseteq D_f^{\alpha}$.

Solution. A function f is not α'-continuous at a point $x \in \mathbf{R}$ if for all $\delta > 0$ there exist $y, z \in (x - \delta, x + \delta)$ such that $|f(y) - f(z)| \geq \alpha' > \alpha$; it follows that f is not α-continuous at x.

Exercise 4.6.10. Let $\alpha > 0$ be given. Show that if f is continuous at x, then it is α-continuous at x as well. Explain how it follows that $D_f^{\alpha} \subseteq D_f$.

Solution. Because f is continuous at x there is a $\delta > 0$ such that

$$y \in (x - \delta, x + \delta) \quad \Rightarrow \quad |f(y) - f(x)| < \frac{\alpha}{2}.$$

If $y, z \in (x - \delta, x + \delta)$ then

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(z) - f(x)| < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Thus f is α-continuous at x. The contrapositive of this result states that if f is not α-continuous at x then f is not continuous at x. It follows that $D_f^{\alpha} \subseteq D_f$.

Exercise 4.6.11. Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n},$$

where $\alpha_n = 1/n$.

Solution. If f is not continuous at x then there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there is a $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| \geq \varepsilon$. It follows that f is not α-continuous at x, where we take $\alpha = \varepsilon$.

Suppose $x \in D_f$. As we just showed, there exists an $\alpha > 0$ such that $x \in D_f^{\alpha}$. Let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \alpha$. We then have $D_f^{\alpha} \subseteq D_f^{\alpha_n}$ (Exercise 4.6.9) and so $x \in D_f^{\alpha_n}$. It follows that

$$D_f \subseteq \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

For the reverse inclusion, note that for each $n \in \mathbf{N}$ we have $D_f^{\alpha_n} \subseteq D_f$ by Exercise 4.6.10. We may conclude that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

Chapter 5. The Derivative

5.2. Derivatives and the Intermediate Value Property

Exercise 5.2.1. Supply proofs for parts (i) and (ii) of Theorem 5.2.4.

Solution.

(i) Observe that

$$\lim_{x\to c}\frac{(f+g)(x)-(f+g)(c)}{x-c}=\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}\right)=f'(c)+g'(c),$$

where we have used Corollary 4.2.4 (ii).

(ii) Observe that

$$\lim_{x\to c}\frac{(kf)(x)-(kf)(c)}{x-c}=\lim_{x\to c}k\left(\frac{f(x)-f(c)}{x-c}\right)=kf'(c),$$

where we have used Corollary 4.2.4 (i).

Exercise 5.2.2. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbf{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

(c) A function f not differentiable at zero and a function g differentiable at zero where $f+g$ is differentiable at zero.

(d) A function f differentiable at zero but not differentiable at any other point.

Solution.

(a) Let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x)=g(x)=\begin{cases}-1 & \text{if } x<0,\\ 1 & \text{if } x\geq 0.\end{cases}$$

Notice that f and g are not continuous at zero and hence not differentiable at zero (Theorem 5.2.3), however the product fg is given by $(fg)(x) = 1$ for all $x \in \mathbf{R}$, which is differentiable everywhere.

(b) Let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

and $g(x) = 0$ for all $x \in \mathbf{R}$. Notice that f is not continuous at zero and hence not differentiable at zero (Theorem 5.2.3), however we have $(fg)(x) = g(x) = 0$ for all $x \in \mathbf{R}$, which is differentiable everywhere.

(c) This is impossible. If g and $f + g$ are differentiable at zero then $f = f + g - g$ must be differentiable at zero by Theorem 5.2.4.

(d) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \in \mathbf{I}. \end{cases}$$

This function is only continuous at zero and hence fails to be differentiable at each nonzero point. We claim that $f'(0) = 0$, i.e.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = 0.$$

Indeed, for any $\varepsilon > 0$ let $\delta = \varepsilon$ and suppose that $x \in \mathbf{R}$ satisfies $0 < |x| < \delta$. Observe that

$$x \in \mathbf{I} \quad \Rightarrow \quad \left| \frac{f(x)}{x} \right| = 0 < \varepsilon, \quad \text{and} \quad x \in \mathbf{Q} \quad \Rightarrow \quad \left| \frac{f(x)}{x} \right| = |x| < \delta = \varepsilon.$$

Thus $f'(0) = 0$.

Exercise 5.2.3.

(a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

(b) Combine the result of part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Solution.

(a) Suppose $x \neq 0$ and observe that

$$h'(x) = \lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} \left[\left(\frac{1}{t} - \frac{1}{x} \right) \frac{1}{t - x} \right]$$
$$= \lim_{t \to x} \left[\left(\frac{x - t}{tx} \right) \frac{1}{t - x} \right] = \lim_{t \to x} \frac{-1}{tx} = \frac{-1}{x^2},$$

where we have used Corollary 4.2.4 (iv).

(b) Keeping the definition of h from part (a), note that $\frac{f(x)}{g(x)} = f(x)h(g(x))$ for any x such that $g(x) \neq 0$. It follows from Theorem 5.2.4 (iii) and the Chain Rule (Theorem 5.2.5) that

$$(f/g)'(x) = f'(x)h(g(x)) + f(x)h'(g(x))g'(x).$$

We can use the result from part (a) to rewrite this as

$$(f/g)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

(c) Suppose $x \in \mathbf{R}$ is such that $g(x) \neq 0$. For any $t \neq x$ (and such that $g(t) \neq 0$; since $g(x) \neq 0$, the continuity of g at x (Theorem 5.2.3) implies that there is some neighbourhood of x where g is non-zero), we have

$$\begin{aligned}\frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} &= \frac{f(t)g(x) - f(x)g(t)}{(t-x)[g(t)g(x)]} \\ &= \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{(t-x)[g(t)g(x)]} \\ &= \frac{f(t)-f(x)}{t-x}\frac{g(x)}{g(t)g(x)} - \frac{g(t)-g(x)}{t-x}\frac{f(x)}{g(t)g(x)}.\end{aligned}$$

It follows that

$$\begin{aligned}(f/g)'(x) &= \lim_{t\to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} \\ &= \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}\left(\frac{g(x)}{g(t)g(x)}\right) - \lim_{t\to x}\left(\frac{g(t)-g(x)}{t-x}\right)\lim_{t\to x}\left(\frac{f(x)}{g(t)g(x)}\right) \\ &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2},\end{aligned}$$

where we have used that f and g are differentiable at x, the continuity of g at x, and several algebraic properties of functional limits (Corollary 4.2.4).

Exercise 5.2.4. Follow these steps to provide a slightly modified proof of the Chain Rule.

(a) Show that a function $h : A \to \mathbf{R}$ is differentiable at $a \in A$ if and only if there exists a function $l : A \to \mathbf{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = l(x)(x-a) \quad \text{for all } x \in A.$$

(b) Use this criterion for differentiability (in both directions) to prove Theorem 5.2.5.

Solution.

(a) Suppose there exists such a function $l : A \to \mathbf{R}$, so that for all $x \in A$ such that $x \neq a$ we have

$$\frac{h(x) - h(a)}{x - a} = l(x).$$

Because l is continuous at a, it follows that $h'(a)$ exists and that $h'(a) = l(a)$.

Now suppose that $h : A \to \mathbf{R}$ is differentiable at a. Define $l : A \to \mathbf{R}$ by

$$l(x) = \begin{cases} \frac{h(x)-h(a)}{x-a} & \text{if } x \neq a, \\ h'(a) & \text{if } x = a. \end{cases}$$

Notice that l satisfies $h(x) - h(a) = l(x)(x - a)$ for all $x \in A$. Furthermore, l is continuous at a:

$$\lim_{x \to a} l(x) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = h'(a) = l(a).$$

(b) Suppose $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ are functions such that $f(A) \subseteq B$, so that the composition $g \circ f : A \to \mathbf{R}$ is defined. Suppose f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$. By part (a) there exist functions $l : A \to \mathbf{R}$ and $L : B \to \mathbf{R}$ such that l is continuous at c, L is continuous at $f(c)$, and

$$f(x) - f(c) = l(x)(x - a) \quad \text{for all } x \in A,$$
$$g(y) - g(f(c)) = L(y)(y - f(c)) \quad \text{for all } y \in B.$$

In particular we have for all $x \in A$,

$$g(f(x)) - g(f(c)) = L(f(x))(f(x) - f(c)) = L(f(x))l(x)(x - a).$$

Since f is differentiable at c it is also continuous at c (Theorem 5.2.3), and since L is continuous at $f(c)$ the composition $L \circ f$ is continuous at c (Theorem 4.3.9). Thus the product $(L \circ f)l$ is continuous at c by Theorem 4.3.4 (iii). It follows from part (a) that $g \circ f$ is differentiable at c and furthermore that

$$(g \circ f)'(c) = L(f(c))l(c) = g'(f(c))f'(c).$$

Exercise 5.2.5. Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

(c) For which values of a is f twice-differentiable?

Solution.

(a) For $a > 0$ we have $\lim_{x\to 0} f_a(x) = 0 = f_a(0)$ and thus f_a is continuous at zero. For $a = 0$ we have

$$\lim_{x\to 0^+} f_a(x) = 1 \neq 0 = \lim_{x\to 0^-} f_a(x)$$

and thus f_a is not continuous at zero. For $a < 0$ we have

$$\lim_{x\to 0^+} f_a(x) = +\infty \neq 0 = \lim_{x\to 0^-} f_a(x)$$

and thus f_a is not continuous at zero. We may conclude that f_a is continuous at zero if and only if $a > 0$.

(b) As we showed in part (a), f_a is not continuous, and hence not differentiable, at zero for $a \leq 0$. For $0 < a < 1$, observe that

$$\lim_{x\to 0^+} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x\to 0^+} x^{a-1} = +\infty \neq 0 = \lim_{x\to 0^-} x^{a-1} = \lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x - 0}.$$

Thus f_a is not differentiable at zero. For $a = 1$ we have

$$\lim_{x\to 0^+} \frac{f_a(x) - f_a(0)}{x - 0} = 1 \neq 0 = \lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x - 0}$$

and thus f_a is not differentiable at zero. For $a > 1$ we have

$$\lim_{x\to 0^+} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x\to 0^+} x^{a-1} = 0 = \lim_{x\to 0^-} x^{a-1} = \lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x - 0}$$

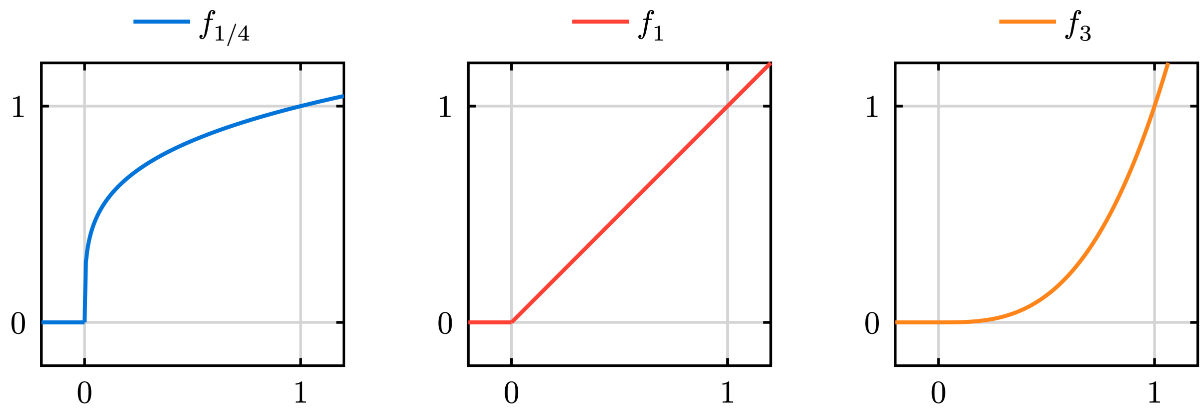
and thus $f_a'(0) = 0$. The derivative function $f_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$f_a'(x) = \begin{cases} ax^{a-1} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

which is continuous since $a > 1$.

(c) Similarly to part (b), f_a is twice-differentiable if and only if $a > 2$, and the second derivative $f_a'' : \mathbf{R} \to \mathbf{R}$ is given by

$$f_a''(x) = \begin{cases} a(a-1)x^{a-2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$



Exercise 5.2.6. Let g be defined on an interval A, and let $c \in A$.

(a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

(b) Assume A is open. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

Solution.

(a) By taking $x = c + h$ or $h = x - c$, we see that the limits

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \quad \text{and} \quad \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}$$

either both diverge or both converge; if they both converge then they must converge to the same value.

(b) Let $\varepsilon > 0$ be given. By part (a) there is a $\delta > 0$ such that

$$0 < |h| < \delta \quad \Rightarrow \quad \left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| < \varepsilon.$$

Note that since $|-h| = |h|$ we also have

$$0 < |h| < \delta \quad \Rightarrow \quad \left| \frac{g(c-h) - g(c)}{-h} - g'(c) \right| < \varepsilon.$$

For any h such that $0 < |h| < \delta$ it follows that

$$\begin{aligned}
\left| \frac{g(c+h) - g(c-h)}{2h} - g'(c) \right| &= \left| \frac{g(c+h) - g(c) + g(c) - g(c-h)}{2h} - \frac{2g'(c)}{2} \right| \\
&\leq \frac{1}{2} \left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| + \frac{1}{2} \left| \frac{g(c-h) - g(c)}{-h} - g'(c) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus

$$\lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h} = g'(c).$$

Exercise 5.2.7. Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbf{R}$ but such that g_a' is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbf{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbf{R}$ and g_a' is differentiable on $\mathbf{R}$, but such that g_a'' is not continuous at zero.

Solution.

(a) Take $a = \frac{5}{3}$. For $x \neq 0$ we have by the usual rules of differentiation that

$$g_a'(x) = \frac{5x\sin\left(\frac{1}{x}\right) - 3\cos\left(\frac{1}{x}\right)}{3\sqrt[3]{x}},$$

and for $x = 0$ we have

$$g_a'(0) = \lim_{t\to 0} \frac{g_a(t)}{t} = \lim_{t\to 0} t^{2/3}\sin\left(\tfrac{1}{t}\right).$$

Since $-t^{2/3} \leq t^{2/3}\sin\left(\frac{1}{t}\right) \leq t^{2/3}$ for every $t \in \mathbf{R}$, the Squeeze Theorem implies that $g_a'(0) = 0$. Thus the derivative $g_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$g_a'(x) = \begin{cases} \dfrac{5x\sin\left(\frac{1}{x}\right) - 3\cos\left(\frac{1}{x}\right)}{3\sqrt[3]{x}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Consider the sequence (x_n) contained in $[0, 1]$ given by $x_n = \frac{1}{\pi(1+2n)}$ and observe that

$$g_a'(x_n) = \frac{5x_n\sin(\pi(1+2n)) - 3\cos(\pi(1+2n))}{3\sqrt[3]{x_n}} = \frac{1}{\sqrt[3]{x_n}}.$$

It follows that $\lim_{n\to\infty} g_a'(x_n) = +\infty$ since (x_n) is positive and satisfies $\lim_{n\to\infty} x_n = 0$. Thus g_a' is unbounded on $[0, 1]$.

(b) Take $a = 3$. For $x \neq 0$ we have by the usual rules of differentiation that

$$g_a'(x) = 3x^2\sin\left(\tfrac{1}{x}\right) - x\cos\left(\tfrac{1}{x}\right).$$

For $x = 0$ we have

$$g_a'(0) = \lim_{t\to 0} \frac{g_a(t)}{t} = \lim_{t\to 0} t^2\sin\left(\tfrac{1}{t}\right) = 0,$$

where we have used the Squeeze Theorem as in part (a). Thus the derivative $g_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$g_a'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that for $x \neq 0$ the function g_a' is given by sums, products, and compositions of continuous functions and hence is itself continuous. For $x = 0$, the Squeeze Theorem shows that $\lim_{x\to 0} g_a'(x) = 0 = g_a'(0)$ and thus g_a' is continuous everywhere.

To see that g_a' is not differentiable at zero, observe that

$$\lim_{t\to 0} 3t\sin\left(\tfrac{1}{t}\right) = 0 \quad \text{and} \quad \lim_{t\to 0}\cos\left(\tfrac{1}{t}\right) \text{ does not exist.}$$

It follows from Corollary 4.2.4 that

$$\lim_{t\to 0} \frac{3t^2\sin\left(\frac{1}{t}\right) - t\cos\left(\frac{1}{t}\right)}{t} = \lim_{t\to 0}\left(3t\sin\left(\tfrac{1}{t}\right) - \cos\left(\tfrac{1}{t}\right)\right)$$

does not exist, i.e. g_a' is not differentiable at zero.

(c) Take $a = 4$. For $x \neq 0$ we have by the usual rules of differentiation that

$$g_a'(x) = 4x^3\sin\left(\tfrac{1}{x}\right) - x^2\cos\left(\tfrac{1}{x}\right).$$

For $x = 0$ we have

$$g_a'(0) = \lim_{t\to 0} t^3 \sin\left(\tfrac{1}{t}\right) = 0,$$

where we have used the Squeeze Theorem as in parts (a) and (b). The derivative $g_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$g_a'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) - x^2\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For $x \neq 0$ we have by the usual rules of differentiation that

$$g_a''(x) = (12x^2 - 1)\sin\left(\tfrac{1}{x}\right) - 6x\cos\left(\tfrac{1}{x}\right).$$

For $x = 0$ we have by the Squeeze Theorem,

$$g_a''(0) = \lim_{t\to 0}\left(4t^2\sin\left(\tfrac{1}{t}\right) - t\cos\left(\tfrac{1}{t}\right)\right) = 0.$$

Thus the second derivative $g_a'' : \mathbf{R} \to \mathbf{R}$ is given by

$$g_a''(x) = \begin{cases} (12x^2 - 1)\sin\left(\frac{1}{x}\right) - 6x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

To see that g_a'' is not continuous at zero, note that

$$\lim_{x\to 0} 12x^2\sin\left(\tfrac{1}{x}\right) = 0, \quad \lim_{x\to 0} 6x\cos\left(\tfrac{1}{x}\right), \quad \text{and} \quad \lim_{x\to 0}\sin\left(\tfrac{1}{x}\right) \text{ does not exist.}$$

It follows from Corollary 4.2.4 that

$$\lim_{x\to 0} g_a''(x) = \lim_{x\to 0}\big(12x^2\sin(\tfrac{1}{x}) - 6x\cos(\tfrac{1}{x}) - \sin(\tfrac{1}{x})\big)$$

does not exist.

Exercise 5.2.8. Review the definition of uniform continuity (Definition 4.4.4). Given a differentiable function $f : A \to \mathbf{R}$, let's say that f is *uniformly differentiable* on A if, given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \varepsilon \quad \text{whenever } 0 < |x-y| < \delta.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbf{R}$? How about $g(x) = x^3$?

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

(c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

Solution.

(a) f is uniformly differentiable on $\mathbf{R}$. Let $\varepsilon > 0$ be given, let $\delta = \varepsilon$, and suppose $x, y \in \mathbf{R}$ are such that $0 < |x - y| < \delta$. It follows that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| = \left|\frac{x^2-y^2}{x-y} - 2y\right| = |x-y| < \delta = \varepsilon.$$

However, g is not uniformly differentiable on $\mathbf{R}$. To see this, we need to show that there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there exist real numbers x, y such that $0 < |x - y| < \delta$ and

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| \geq \varepsilon.$$

We claim that $\varepsilon = 1$ satisfies the previous statement. Indeed, let $\delta > 0$ be given. Let $x = \frac{2}{\delta}$ and $y = x + \frac{\delta}{2}$, so that $0 < |x - y| = \frac{\delta}{2} < \delta$, and observe that

$$\begin{aligned}\left|\frac{g(x)-g(y)}{x-y} - g'(y)\right| &= \left|\frac{x^3-y^3}{x-y} - 3y^2\right| \\ &= \left|\frac{(x-y)(x^2+xy+y^2)}{x-y} - 3y^2\right| \\ &= \left|x^2+xy-2y^2\right| \\ &= |x-y||x+2y| \\ &= \tfrac{\delta}{2}|3x+\delta| \\ &= \tfrac{\delta}{2}(3x+\delta) = \frac{3x\delta}{2} + \frac{\delta^2}{2} > \frac{x\delta}{2} = 1.\end{aligned}$$

(b) Suppose $f : A \to \mathbf{R}$ is uniformly differentiable. Fix $\varepsilon > 0$. Since f is uniformly differentiable, there exists a $\delta > 0$ such that

$$0 < |s - t| < \delta \quad \Rightarrow \quad \left| \frac{f(s) - f(t)}{s - t} - f'(t) \right| < \frac{\varepsilon}{2}.$$

Fix $y \in A$ and suppose $x \in A$ is such that $0 < |x - y| < \delta$. Observe that

$$|f'(x) - f'(y)| \leq \left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus f' is continuous.

(c) There is no analogous theorem. Consider the function

$$f(x) = \begin{cases} x^{5/3} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

As we showed in Exercise 5.2.7 (a), f is differentiable on $\mathbf{R}$, and hence on $[0, 1]$, but f' is unbounded on $[0, 1]$. It follows that f' is not continuous on $[0, 1]$ (since continuous functions preserve compactness) and hence by part (b) of this exercise, f cannot be uniformly differentiable on $[0, 1]$.

Exercise 5.2.9. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(c) If f is differentiable on an interval containing zero and if $\lim_{x \to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution.

(a) This is true. If $f : I \to \mathbf{R}$ is differentiable and f' is not constant, where I is an interval, then there exist distinct $x, y \in I$ such that $f'(x) \neq f'(y)$; we may assume that $f'(x) < f'(y)$. Darboux's Theorem (Theorem 5.2.7) implies that $[f'(x), f'(y)] \subseteq f'(I)$, from which it follows that f' takes on at least one irrational value in the proper interval $[f'(x), f'(y)]$.

(b) This is false. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

By the usual rules of differentiation and the Squeeze Theorem, the derivative $f' : \mathbf{R} \to \mathbf{R}$ is given by

$$f'(x) = \begin{cases} \frac{1}{2} + 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ \frac{1}{2} & \text{if } x = 0; \end{cases}$$

note that $f'(0) > 0$. Let (x_n) be the sequence given by $x_n = \frac{1}{2\pi n}$, so that $\lim_{n\to\infty} x_n = 0$, and observe that

$$f'(x_n) = -\tfrac{1}{2} < 0$$

for every $n \in \mathbf{N}$. It follows that for every δ-neighbourhood $V_\delta(0)$ we can find some $x_n \in V_\delta(0)$ such that $f(x_n) < 0$.

(c) This is true and we will prove it by contradiction. Suppose that $L > f'(0)$; the case where $L < f'(0)$ is handled similarly. Let $\varepsilon = L - f'(0) > 0$. Since $\lim_{x\to 0} f'(x) = L$, there is a $\delta > 0$ such that

$$x \in (-\delta, \delta) \text{ and } x \neq 0 \quad \Rightarrow \quad f'(x) \in \left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right). \tag{1}$$

In particular we have $f'\left(\frac{\delta}{2}\right) \in \left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)$. Since

$$f'(0) < L - \frac{3\varepsilon}{4} < L - \frac{\varepsilon}{2} < f'\left(\tfrac{\delta}{2}\right),$$

Darboux's Theorem (Theorem 5.2.7) implies that there is a $y \in \left(0, \frac{\delta}{2}\right)$ such that $f'(y) = L - \frac{3\varepsilon}{4}$, which contradicts (1).

Exercise 5.2.10. Recall that a function $f : (a, b) \to \mathbf{R}$ is *increasing* on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbf{R}$ and satisfies $g'(0) > 0$. Now, prove that g is *not* increasing over any open interval containing 0.

In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

Solution. As we showed in Exercise 5.2.9 (b), g is differentiable on $\mathbf{R}$ and satisfies $g'(0) = \frac{1}{2} > 0$. For $n \in \mathbf{N}$ let

$$x_n = \frac{1}{2\pi n} \quad \text{and} \quad y_n = \frac{1}{2\pi n - \frac{\pi}{2}}.$$

Suppose (a, b) is some open interval containing 0 and let $N \in \mathbf{N}$ be such that $y_N < b$, so that $0 < x_N < y_N < b$. Observe that

$$g(x_N) = \frac{1}{4\pi N} \quad \text{and} \quad g(y_N) = \frac{1}{4\pi N - \pi} - \frac{1}{\left(2\pi N - \frac{\pi}{2}\right)^2}.$$

Some algebraic manipulation reveals that

$$g(x_N) - g(y_N) = \frac{\pi(8N-3)-4}{(\pi - 4\pi N)^2} + \frac{\pi}{4N(\pi - 4\pi N)^2} > 0.$$

Thus $x_N, y_N \in (a,b)$ are such that $x_N < y_N$ but $g(x_N) > g(y_N)$; it follows that g is not increasing on (a,b).

Exercise 5.2.11. Assume that g is differentiable on $[a,b]$ and satisfies $g'(a) < 0 < g'(b)$.

(a) Show that there exists a point $x \in (a,b)$ where $g(a) > g(x)$, and a point $y \in (a,b)$ where $g(y) < g(b)$.

(b) Now complete the proof of Darboux's Theorem started earlier.

Solution.

(a) We will prove the contrapositive statement:

$$g(a) \leq g(x) \text{ for all } x \in (a,b) \quad \Rightarrow \quad g'(a) \geq 0.$$

Suppose that for all $x \in (a,b)$ we have $g(a) \leq g(x)$. Let (x_n) be the sequence given by $x_n = a + \frac{b-a}{2n}$, so that $x_n \in (a,b)$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} x_n = a$. It follows from Theorem 4.2.3 that

$$\lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} = g'(a).$$

The denominator of $\frac{g(x_n)-g(a)}{x_n - a}$ is positive for each $n \in \mathbf{N}$ and our assumption that $g(a) \leq g(x)$ for all $x \in (a,b)$ implies that the numerator is non-negative for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) allows us to conclude that $g'(a) \geq 0$.

The existence of a point $y \in (a,b)$ such that $g(y) < g(b)$ can be proved similarly.

(b) The function g is differentiable and hence continuous on $[a,b]$ (Theorem 5.2.3) and thus achieves a minimum value at some $c \in [a,b]$ by the Extreme Value Theorem (Theorem 4.4.2). By part (a) we actually have $c \in (a,b)$ and thus $g'(c) = 0$ by the Interior Extremum Theorem (Theorem 5.2.6).

Exercise 5.2.12 (Inverse functions). If $f : [a, b] \to \mathbf{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8 we saw that if f is continuous on $[a, b]$ then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$. Show f^{-1} is differentiable with

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x).$$

Solution. Since $f'(x) \neq 0$ we have

$$\lim_{s \to x} \frac{s - x}{f(s) - f(x)} = \frac{1}{f'(x)}$$

by Corollary 4.2.4 (iv). Let $\varepsilon > 0$ be given. There is a $\delta_1 > 0$ such that

$$0 < |s - x| < \delta_1 \quad \Rightarrow \quad \left| \frac{s - x}{f(s) - f(x)} - \frac{1}{f'(x)} \right| < \varepsilon. \tag{1}$$

Because f^{-1} is continuous on its domain we have $\lim_{t \to y} f^{-1}(t) = f^{-1}(y) = x$. Thus there exists a $\delta_2 > 0$ such that

$$0 < |t - y| < \delta_2 \quad \Rightarrow \quad 0 < \left| f^{-1}(t) - f^{-1}(y) \right| = \left| f^{-1}(t) - x \right| < \delta_1.$$

(The fact that $f^{-1}(t) \neq x$ follows since $t \neq y$ and f^{-1} is injective.) We may now take $s = f^{-1}(t)$ in (1) to see that

$$0 < |t - y| < \delta_2 \quad \Rightarrow \quad \left| \frac{f^{-1}(t) - x}{f(f^{-1}(t)) - f(x)} - \frac{1}{f'(x)} \right| = \left| \frac{f^{-1}(t) - f^{-1}(y)}{t - y} - \frac{1}{f'(x)} \right| < \varepsilon.$$

Thus

$$\left(f^{-1}\right)'(y) = \lim_{t \to y} \frac{f^{-1}(t) - f^{-1}(y)}{t - y} = \frac{1}{f'(x)}.$$

5.3. The Mean Value Theorems

Exercise 5.3.1. Recall from Exercise 4.4.9 that a function $f : A \to \mathbf{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$ then f is Lipschitz on $[a, b]$.

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Solution.

(a) Note that $|f'|$ is continuous on $[a, b]$ since f' is continuous on $[a, b]$. The Extreme Value Theorem (Theorem 4.4.2) then implies that $|f'|$ attains a maximum on $[a, b]$, say $M = |f'(t)|$ for some $t \in [a, b]$. Let $x < y$ in $[a, b]$ be given. The Mean Value Theorem (Theorem 5.3.2) on the interval $[x, y]$ implies that there is a $c \in (x, y)$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \leq M.$$

Thus f is Lipschitz on $[a, b]$.

(b) If $|f'(x)| < 1$ on $[a, b]$ then the maximum value $M = |f'(t)|$ from part (a) must satisfy $M < 1$ and thus f is contractive on $[a, b]$.

Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Solution. We will prove the contrapositive statement. If there exist points $x < y$ in A such that $f(x) = f(y)$ then Rolle's Theorem implies that there is some $c \in (x, y)$ such that $f'(c) = 0$.

For a counterexample to the converse statement, consider the injective function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$, which satisfies $f'(0) = 0$.

Exercise 5.3.3. Let h be a differentiable function defined on the interval $[0,3]$, and assume that $h(0)=1, h(1)=2$, and $h(3)=2$.

(a) Argue that there exists a point $d \in [0,3]$ where $h(d)=d$.

(b) Argue that at some point c we have $h'(c)=1/3$.

(c) Argue that $h'(x)=1/4$ at some point in the domain.

Solution.

(a) Define $f:[0,3] \to \mathbf{R}$ by $f(x)=h(x)-x$ and note that f is continuous since h is continuous. Furthermore, since $f(1)=h(1)-1=1$ and $f(3)=h(3)-3=-1$, the Intermediate Value Theorem implies that there is some $d \in (1,3)$ such that $f(d)=0$, i.e. $h(d)=d$.

(b) Because h is differentiable on $[0,3]$, the Mean Value Theorem implies that there exists some point $c \in (0,3)$ such that

$$h'(c) = \frac{h(3)-h(0)}{3-0} = \frac{1}{3}.$$

(c) Similarly to part (b), the Mean Value Theorem implies the existence of some $b \in (1,3)$ such that

$$h'(b) = \frac{h(3)-h(1)}{3-1} = 0.$$

Combining this part (b), we see that h' takes the values 0 and $\frac{1}{3}$. Since $0 < \frac{1}{4} < \frac{1}{3}$, Darboux's Theorem implies that h' takes the value $\frac{1}{4}$ at some point in $\left(0, \frac{1}{3}\right) \subseteq [0,3]$.

Exercise 5.3.4. Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

(a) If $f(x_n)=0$ for all $n \in \mathbf{N}$, show $f(0)=0$ and $f'(0)=0$.

(b) Add the assumption that f is twice-differentiable at zero and show that $f''(0)=0$ as well.

Solution.

(a) We have

$$0 = \lim_{n\to\infty} f(x_n) = f(0)$$

since f is continuous at zero.

Note that, since $x_n \neq 0$ for each $n \in \mathbf{N}$, the difference quotient $\frac{f(x_n)-f(0)}{x_n-0} = \frac{f(x_n)}{x_n}$ is well-defined and satisfies $\frac{f(x_n)}{x_n} = 0$. Since $f'(0)$ exists, Theorem 4.2.3 implies that

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0.$$

(b) We are given that the limit

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{f'(x)}{x}$$

exists. Since $f(0) = 0$, we may apply L'Hospital's Rule (Theorem 5.3.6) to see that

$$f''(0) = \lim_{x\to 0} \frac{2f(x)}{x^2}.$$

Since $x_n \neq 0$ for each $n \in \mathbf{N}$, the quotient $\frac{2f(x_n)}{x_n^2}$ is well-defined and satisfies $\frac{2f(x_n)}{x_n^2} = 0$. Since $\lim_{x\to 0} \frac{2f(x)}{x^2}$ exists, Theorem 4.2.3 gives us

$$f''(0) = \lim_{n\to\infty} \frac{2f(x_n)}{x_n^2} = 0.$$

Exercise 5.3.5.

(a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem (Theorem 5.3.5).

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve.)

Solution.

(a) Define $h : [a, b] \to \mathbf{R}$ by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

and note that h is continuous on $[a, b]$ and differentiable on (a, b). Note further that $h(b) - h(a) = 0$. The Mean Value Theorem (Theorem 5.3.2) implies the existence of some $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a},$$

or equivalently

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = \frac{h(b) - h(a)}{b - a} = 0.$$

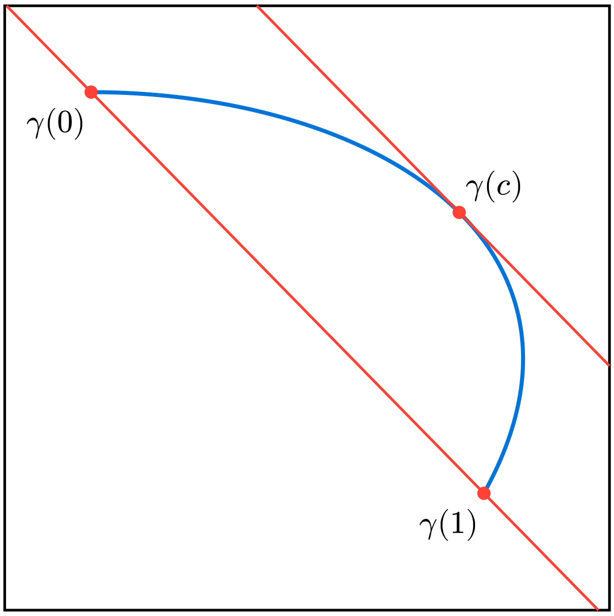
(b) If $f'(c) \neq 0$ and $f(b) \neq f(a)$, so that

$$\frac{g'(c)}{f'(c)} = \frac{g(b) - f(a)}{f(b) - f(a)},$$

then the Generalized Mean Value Theorem can be geometrically interpreted as asserting the existence of a tangent line to the planar curve $\gamma : [a, b] \to \mathbf{R}^2$ given by

$\gamma(t) = (f(t), g(t))$ at the point $\gamma(c) = (f(c), g(c))$ which is parallel to the line through the points $\gamma(a) = (f(a), g(a))$ and $\gamma(b) = (f(b), g(b))$.

$\gamma(t) = \left(\sin 2t, \cos \frac{3}{2}t\right)$ for $t \in [0, 1]$



Exercise 5.3.6.

(a) Let $g : [0, a] \to \mathbf{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show $|g(x)| \leq Mx$ for all $x \in [0, a]$.

(b) Let $h : [0, a] \to \mathbf{R}$ be twice differentiable $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.

(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Solution.

(a) The inequality $|g(x)| \leq Mx$ is clear when $x = 0$, since $g(0) = 0$. Suppose $x \in (0, a]$. By the Mean Value Theorem on the interval $[0, x]$, there exists some $c \in (0, x)$ such that

$$|g'(c)| = \left|\frac{g(x)}{x}\right| \quad \Rightarrow \quad |g(x)| = |g'(c)|x \leq Mx.$$

(b) The inequality $|h(x)| \leq \frac{1}{2}Mx^2$ is clear when $x = 0$, since $h(0) = 0$. Suppose $x \in (0, a]$. Using the Generalized Mean Value Theorem on the interval $[0, x]$ with the functions h and $\frac{1}{2}x^2$, we can find some $c \in (0, x)$ such that

$$\frac{h(x)}{\frac{1}{2}x^2} = \frac{h'(c)}{c}.$$

Now we can use the Mean Value Theorem on the interval $[0, c]$ with the function h' to obtain some $d \in (0, c)$ such that

$$h''(d) = \frac{h'(c)}{c}.$$

Combining this with the previous equality, we see that

$$h''(d) = \frac{h(x)}{\frac{1}{2}x^2} \quad \Rightarrow \quad |h(x)| = \tfrac{1}{2}h''(d)x^2 \leq \tfrac{1}{2}Mx^2.$$

(c) Suppose $f : [0, a] \to \mathbf{R}$ is three times differentiable, $f''(0) = f'(0) = f(0) = 0$, and $f'''(x) \leq M$ for all $x \in [0, a]$. We claim that $|f(x)| \leq \frac{1}{6}Mx^3$ for all $x \in [0, a]$.

The inequality $|f(x)| \leq \frac{1}{6}Mx^3$ is clear when $x = 0$, since $f(0) = 0$. Suppose $x \in (0, a]$. Using the Generalized Mean Value Theorem on the interval $[0, x]$ with the functions f and $\frac{1}{6}x^3$, we can find some $b \in (0, x)$ such that

$$\frac{f(x)}{\frac{1}{6}x^3} = \frac{f'(b)}{\frac{1}{2}b^2}.$$

Using the Generalized Mean Value Theorem on the interval $[0, b]$ with the functions f' and $\frac{1}{2}x^2$, we can find some $c \in (0, b)$ such that

$$\frac{f'(b)}{\frac{1}{2}b^2} = \frac{f''(c)}{c}.$$

Now we can use the Mean Value Theorem on the interval $[0, c]$ with the function f'' to find some $d \in (0, c)$ such that

$$f'''(d) = \frac{f''(c)}{c}.$$

Combining all of these equalities, we see that

$$f'''(d) = \frac{f(x)}{\frac{1}{6}x^3} \quad \Rightarrow \quad |f(x)| = \tfrac{1}{6}|f'''(d)|x^3 \leq \tfrac{1}{6}Mx^3.$$

Exercise 5.3.7. A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Solution. We will prove the contrapositive statement. Suppose that $x < y$ belong to the domain of f and are such that $f(x) = x$ and $f(y) = y$. By the Mean Value Theorem on the interval $[x, y]$, there exists some $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{x - y} = \frac{y - x}{y - x} = 1.$$

Exercise 5.3.8. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x \to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Solution. We would like to see that the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

exists and equals L. Letting I denote the interval domain of f, note that the numerator and denominator of this fraction are both continuous on I, differentiable on $I \setminus \{0\}$, and vanish at zero. Thus we have satisfied the hypotheses of the 0/0 case of L'Hospital's Rule. Applying the rule, we find that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} f'(x) = L.$$

Exercise 5.3.9. Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the 0/0 case of L'Hospital's Rule under this stronger hypothesis.

Solution. Note that for all $x \neq a$ we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

By assumption the limits

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a}$$

both exist and $g'(a) \neq 0$. It follows from Corollary 4.2.4 (iv) that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

Now we can use our assumption that f' and g' are continuous at a with $g'(a) \neq 0$ to see that

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

Exercise 5.3.10. Let $f(x) = x\sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x), g(x), f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of Theorem 5.3.6.

Solution. Some algebraic manipulation reveals that

$$\frac{f(x)}{g(x)} = x\sin\Big(\frac{1}{x^4}\Big) \quad \text{and} \quad \frac{f'(x)}{g'(x)} = \sin\Big(\frac{1}{x^4}\Big)\Big(\frac{x^3}{2} + x\Big) - \frac{2\cos\big(\frac{1}{x^4}\big)}{x}.$$

Given an $\varepsilon > 0$, notice that

$$0 < |x| < \sqrt{\frac{1}{\log\left(\frac{1}{\varepsilon}\right)}} \quad \Rightarrow \quad e^{-1/x^2} < \varepsilon.$$

Thus $\lim_{x\to 0} g(x) = 0$. Combining this with various applications of the Squeeze Theorem, we find that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{g(x)} = 0.$$

However, we claim that $\frac{f'(x)}{g'(x)}$ does not converge to zero as $x \to 0$. Indeed, consider the sequence (x_n) given by

$$x_n = \frac{1}{\sqrt[4]{2n\pi}} \to 0 \text{ as } n \to \infty$$

and observe that

$$\frac{f'(x_n)}{g'(x_n)} = -2\sqrt[4]{2n\pi} \to -\infty \text{ as } n \to \infty.$$

This does not conflict with the content of Theorem 5.3.6, which states that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = L \quad \Rightarrow \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = L,$$

and does *not* state that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = L \quad \Rightarrow \quad \lim_{x\to 0} \frac{f'(x)}{g'(x)} = L.$$

Exercise 5.3.11.

(a) Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hospital's Rule (Theorem 5.3.6).

(b) If we keep the first part of the hypothesis of Theorem 5.3.6 the same but we assume that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty,$$

does it necessarily follow that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \infty?$$

Solution.

(a) Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad \left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon.$$

Suppose $x \in (a-\delta, a)$. By the Generalized Mean Value Theorem on the interval $[x, a]$, there exists some $c \in (x, a)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a)-f(x)}{g(a)-g(x)} = \frac{f(x)}{g(x)};$$

note we are using that g' does not vanish on (x, a). Since $c \in (a-\delta, a)$, we then have

$$0 < |c-a| < \delta \quad \Rightarrow \quad \left|\frac{f'(c)}{g'(c)} - L\right| = \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon.$$

We can similarly handle the case where $x \in (a, a+\delta)$ by using the Generalized Mean Value Theorem on the interval $[a, x]$. In any case, we have shown that

$$0 < |x-a| < \delta \quad \Rightarrow \quad \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon.$$

It follows that $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

(b) It does necessarily follow; the proof from part (a) needs only slight modifications. Let $M > 0$ be given. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$, there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad \frac{f'(x)}{g'(x)} \geq M.$$

Suppose $x \in (a-\delta, a)$. By the Generalized Mean Value Theorem on the interval $[x, a]$ there exists some $c \in (x, a)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a)-f(x)}{g(a)-g(x)} = \frac{f(x)}{g(x)};$$

note we are using that g' does not vanish on (x, a). Since $c \in (a-\delta, a)$, we then have

$$0 < |c-a| < \delta \quad \Rightarrow \quad \frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)} \geq M.$$

We can similarly handle the case where $x \in (a, a+\delta)$ by using the Generalized Mean Value Theorem on the interval $[a, x]$. In any case, we have shown that

$$0 < |x-a| < \delta \quad \Rightarrow \quad \frac{f(x)}{g(x)} \geq M.$$

It follows that $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$.

Exercise 5.3.12. If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

(Compare this to Exercise 5.2.6(b).)

Solution. Using the 0/0 case of L'Hospital's Rule, we find that

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h}.$$

Since f' is differentiable at a we may apply Exercise 5.2.6 (b) to see that

$$\lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} = f''(a).$$

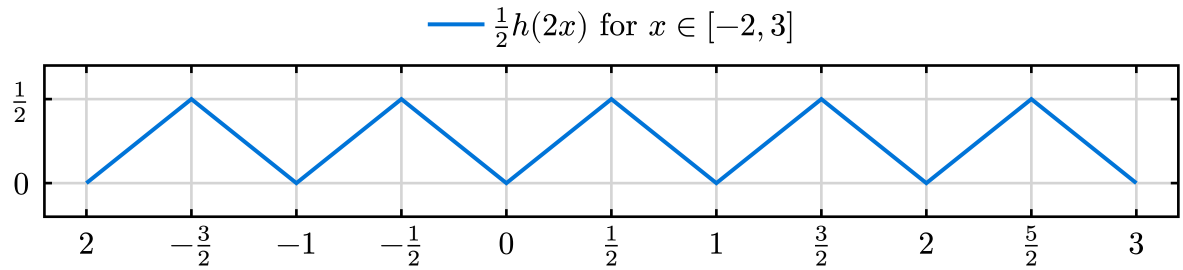
5.4. A Continuous Nowhere-Differentiable Function

Exercise 5.4.1. Sketch a graph of $(1/2)h(2x)$ on $[-2, 3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n} h(2^n x)$$

as n gets larger.

Solution. Each h_n is a periodic "sawtooth" function; as n gets larger, the "teeth" get more densely packed and the peaks get lower.



Exercise 5.4.2. Fix $x \in \mathbf{R}$. Argue that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

converges and thus $g(x)$ is properly defined.

Solution. Since $0 \le h(x) \le 1$ we have $0 \le 2^{-n} h(2^n x) \le 2^{-n}$ for each $n \in \mathbf{N}$. As the series $\sum_{n=0}^{\infty} 2^{-n}$ is convergent (Example 2.7.5), the series $\sum_{n=0}^{\infty} 2^{-n} h(2^n x)$ is also convergent by the Comparison Test (Theorem 2.7.4).

Exercise 5.4.3. Taking the continuity of $h(x)$ as given, reference the proper theorems from Chapter 4 that imply that the *finite* sum

$$g_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} h(2^n x)$$

is continuous on $\mathbf{R}$.

Solution. The continuity of g_m follows from Theorem 4.3.4 and Theorem 4.3.9.

Exercise 5.4.4. As the graph of Figure 5.7 suggests, the structure of $g(x)$ is quite intricate. Answer the following questions, assuming that $g(x)$ is indeed continuous.

(a) How do we know g attains a maximum value M on $[0, 2]$? What is this value?

(b) Let D be the set of points in $[0, 2]$ where g attains its maximum. That is $D = \{x \in [0, 2] : g(x) = M\}$. Find one point in D.

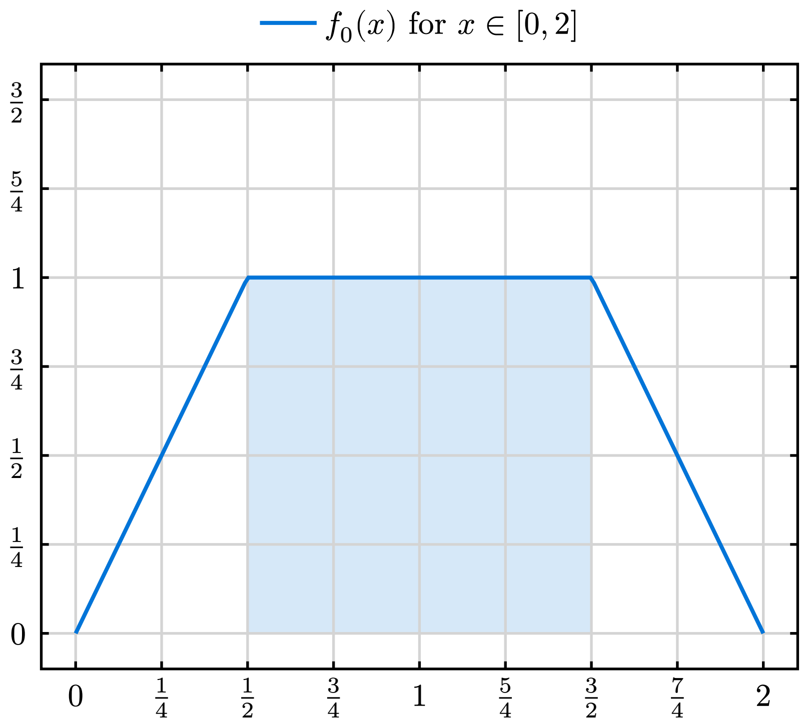
(c) Is D finite, countable, or uncountable?

Solution.

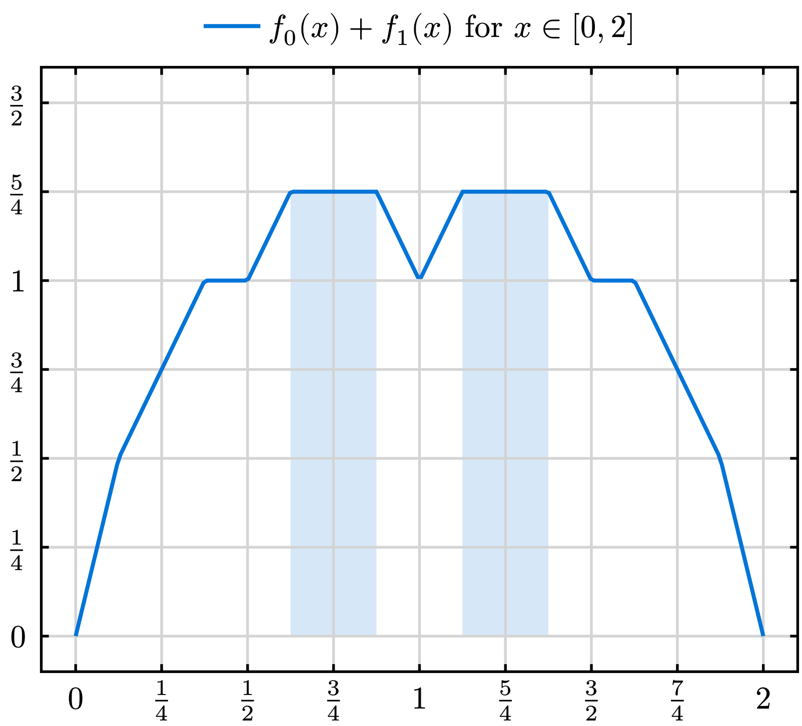
(a) Since g is continuous on the compact set $[0, 2]$, we know it attains a maximum here by the Extreme Value Theorem (Theorem 4.4.2). To find this maximum value M, for each non-negative integer n let $f_n(x) = 2^{-2n}h(2^{2n}x) + 2^{-2n-1}h(2^{2n+1}x)$, so that

$$f_0(x) = h(x) + \tfrac{1}{2}h(2x), \quad f_1(x) = \tfrac{1}{4}h(4x) + \tfrac{1}{8}h(8x), \quad \text{etc.}$$

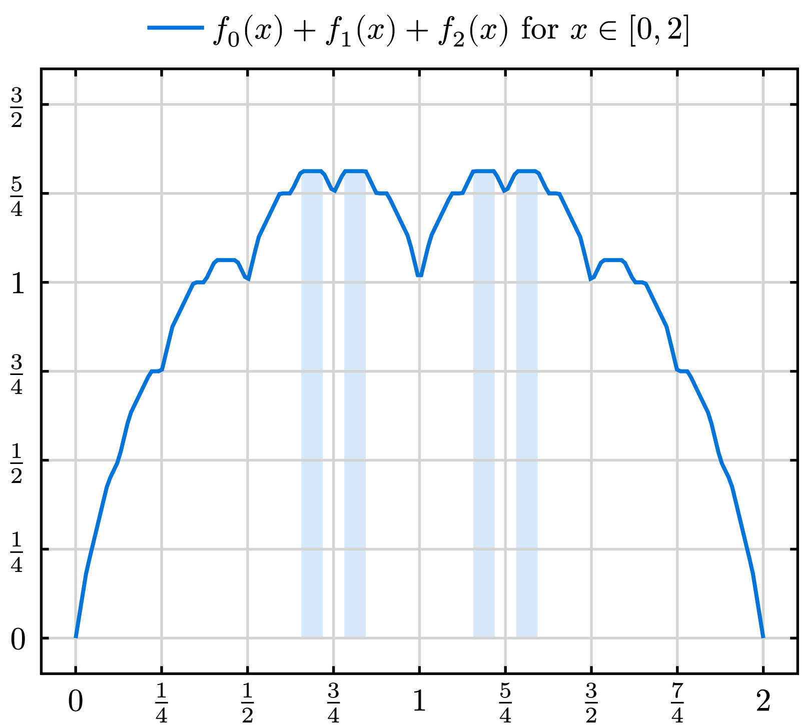
Thus $g(x) = h(x) + \frac{1}{2}h(2x) + \frac{1}{4}h(4x) + \frac{1}{8}h(8x) + \cdots = f_0(x) + f_1(x) + \cdots$; for any given x such a regrouping of terms is justified by Exercise 2.5.3. Here is a graph of f_0 on $[0, 2]$.



Note that $f_0(x) = 1$ on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. Note further that $f_1(x) = \frac{1}{4}f_0(4x)$, so that on the interval $[0, 2]$ the function f_1 is given by four copies of f_0 scaled by a factor of $\frac{1}{4}$. The interval $\left[\frac{1}{2}, \frac{3}{2}\right]$, where f_0 is constant, contains two of the intervals of length $\frac{1}{4}$ where f_1 is also constant; see the following figure.



On these intervals we then have $f_0(x) = f_1(x) = 1 + \frac{1}{4}$. Similarly, f_2 is given by $f_2(x) = \frac{1}{16} f_0(16x)$, and there are further subintervals of the previous subintervals where f_2 is also constant. On these subintervals we have $f_0(x) + f_1(x) + f_2(x) = 1 + \frac{1}{4} + \frac{1}{16}$; see the following figure.



We can continue arguing in this manner to see that $M \geq 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3}$. On the other hand, since each f_n satisfies $f_n(x) \leq 4^{-n}$ on $[0, 2]$, for each $x \in [0, 2]$ we have

$$g(x) = f_0(x) + f_1(x) + f_2(x) + \cdots \leq 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3}.$$

We may conclude that $M = \frac{4}{3}$.

(b) First, we claim that for every non-negative integer n we have $h\left(\frac{2^{n+1}}{3}\right) = \frac{2}{3}$. The base case $n = 0$ is clear, so suppose that the result is true for some n and observe that

$$h\left(\frac{2^{n+2}}{3}\right) = h\left(\frac{2^{n+2}}{3} - 2^{n+1}\right) = h\left(\frac{2^{n+2}(2-3)}{3}\right) = h\left(-\frac{2^{n+1}}{3}\right) = h\left(\frac{2^{n+1}}{3}\right) = \frac{2}{3},$$

where we have used our induction hypothesis and that h is an even 2-periodic function. Our claim follows by induction.

Now observe that

$$g\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} h\left(\frac{2^{n+1}}{3}\right) = \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{4}{3} = M.$$

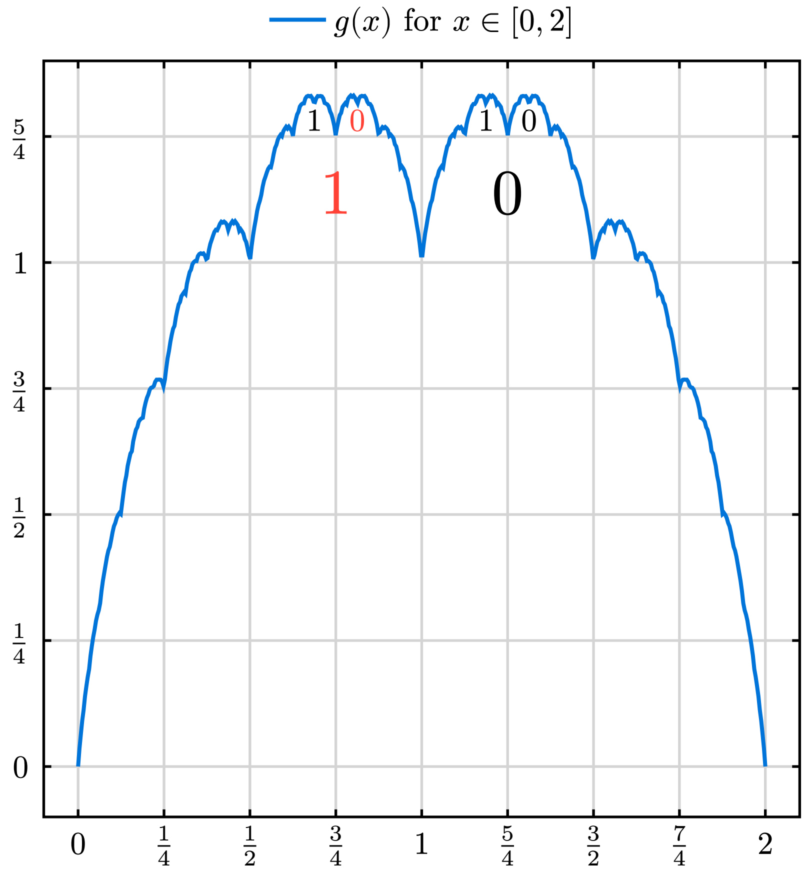
Thus $\frac{2}{3} \in D$.

(c) D is uncountable; we will show that D is in bijection with $\mathbf{R}$. First, we will inject the collection of binary sequences into D.

Let $b : \mathbf{N} \to \{0, 1\}$ be a binary sequence. We claim that

$$x_b := 1 + \sum_{k=1}^{\infty} \frac{(-1)^{b(k)}}{4^k}$$

belongs to D, i.e. satisfies $x_b \in [0, 2]$ and $g(x_b) = M = \frac{4}{3}$. For intuition, consider the following graph of g on $[0, 2]$. The binary sequence tells us whether to go left or right at each "bump". For example, if $b(0) = 1$ and $b(1) = 0$ then we should follow the red "path".



First, observe that

$$\frac{2}{3} \le 1 - \sum_{k=1}^{\infty} \frac{1}{4^k} \le x_b \le 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{4}{3}.$$

Thus $x_b \in [0, 2]$. To show that $g(x_b) = \frac{4}{3}$, let us express g as

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{4^n} f_0(4^n x).$$

(See part (a) for the definition of f_0, the justification for this expression, and a graph of f_0 on the interval $[0, 2]$.) Suppose that K is a non-negative integer and $n \ge K + 1$. Observe that

$$4^n \left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K} \right)$$

is an even integer and thus by the 2-periodicity of f_0 we have

$$f_0\left(4^n \left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K} \right) \right) = f_0(0) = 0.$$

Furthermore, observe that

$$4^K \left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K} \right)$$

is an odd integer and thus by the 2-periodicity of f_0 we have

$$f_0\left(4^K \left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K} \right) \right) = f_0(1) = 1.$$

If $K \ge 1$, suppose that $0 \le n \le K - 1$ and observe that

$$\begin{aligned} &4^n \left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K} \right) \\ &\qquad = \underbrace{4^n + (-1)^{b(1)} 4^{n-1} + \cdots + (-1)^{b(n)}}_{\text{odd integer}} + \frac{(-1)^{b(n+1)}}{4} + \cdots + \frac{(-1)^{b(k)}}{4^{K-n}}. \end{aligned}$$

It follows from the 2-periodicity of f_0 that

$$f_0\left(4^n \left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K} \right) \right) = f_0\left(1 + \frac{(-1)^{b(n+1)}}{4} + \cdots + \frac{(-1)^{b(k)}}{4^{K-n}} \right).$$

Note that

$$\frac{2}{3} \le 1 - \sum_{k=1}^{\infty} \frac{1}{4^k} \le 1 + \frac{(-1)^{b(n+1)}}{4} + \cdots + \frac{(-1)^{b(k)}}{4^{K-n}} \le 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{4}{3}.$$

Since $f_0(x) = 1$ on the interval $\left[\frac{2}{3}, \frac{4}{3}\right]$, it follows that

$$f_0\left(1 + \frac{(-1)^{b(n+1)}}{4} + \cdots + \frac{(-1)^{b(k)}}{4^{K-n}}\right) = 1.$$

To summarize our findings, for each non-negative integer K we have

$$f_0\left(4^n\left(1 + \sum_{k=1}^{K} \frac{(-1)^{b(k)}}{4^k}\right)\right) = \begin{cases} 1 & \text{if } 0 \le n \le K, \\ 0 & \text{if } n > K. \end{cases} \tag{1}$$

We can now show that $g(x_b) = \frac{4}{3}$:

$$\begin{aligned} g(x_b) &= g\left(1 + \lim_{K\to\infty} \sum_{k=1}^{K} \frac{(-1)^{b(k)}}{4^k}\right) \\ &= \lim_{K\to\infty} g\left(1 + \sum_{k=1}^{K} \frac{(-1)^{b(k)}}{4^k}\right) \\ &= \lim_{K\to\infty} \sum_{n=0}^{\infty} \frac{1}{4^n} f_0\left(4^n\left(1 + \sum_{k=1}^{K} \frac{(-1)^{b(k)}}{4^k}\right)\right) \\ &= \lim_{K\to\infty} \sum_{n=0}^{K} \frac{1}{4^n} \\ &= \frac{4}{3}; \end{aligned}$$

note we have used the continuity of g at x_b for the second equality and equation (1) for the fourth equality. If we let S be the collection of binary sequences

$$S = \{b : \mathbf{N} \to \{0, 1\}\}$$

and define $\Psi : S \to D$ by $\Psi(b) = x_b$, then we have shown that Ψ is well-defined. Now we will show that Ψ is injective. Suppose that $a, b \in S$ are such that $a \neq b$ and let

$$K \coloneqq \min\{k \in \mathbf{N} : a(k) \neq b(k)\};$$

without loss of generality, we may assume that $a(K) = 1$ and $b(K) = 0$. Thus

$$x_a = 1 + \frac{(-1)^{a(1)}}{4} + \frac{(-1)^{a(2)}}{16} + \cdots - \frac{1}{4^K} + \frac{(-1)^{a(K+1)}}{4^{K+1}} + \cdots,$$

$$x_b = 1 + \frac{(-1)^{b(1)}}{4} + \frac{(-1)^{b(2)}}{16} + \cdots + \frac{1}{4^K} + \frac{(-1)^{b(K+1)}}{4^{K+1}} + \cdots.$$

It follows that

$$x_b - x_a = \frac{2}{4^K} + \frac{(-1)^{b(K+1)} - (-1)^{a(K+1)}}{4^{K+1}} + \frac{(-1)^{b(K+2)} - (-1)^{a(K+2)}}{4^{K+2}} + \cdots$$

and hence that

$$4^{-K}(x_b - x_a) - 2 = \frac{(-1)^{b(K+1)} - (-1)^{a(K+1)}}{4} + \frac{(-1)^{b(K+2)} - (-1)^{a(K+2)}}{16} + \cdots$$

$$\geq -2\left(\frac{1}{4} + \frac{1}{16} + \cdots\right)$$

$$= -\frac{2}{3}.$$

Thus $4^{-K}(x_b - x_a) \geq \frac{4}{3} > 0$, whence $x_b > x_a$. It follows that Ψ is injective.

As we showed in Exercise 1.6.9, S is in bijection with $\mathbf{R}$; composing this bijection with Ψ gives us an injection $\mathbf{R} \to D$. Certainly the inclusion $D \hookrightarrow \mathbf{R}$ is an injection and thus, by the Schröder-Bernstein Theorem (Exercise 1.5.11), we may conclude that D is in bijection with $\mathbf{R}$.

Exercise 5.4.5. Show that

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1,$$

and use this to prove that $g'(0)$ does not exist.

Solution. For any $m \in \{0, 1, 2, \ldots\}$ we have

$$h(2^{n-m}) = \begin{cases} 2^{n-m} & \text{if } 0 \leq n \leq m, \\ 0 & \text{if } n > m; \end{cases}$$

if $0 \leq n \leq m$ we have $0 < 2^{n-m} \leq 1$ and if $n > m$ then 2^{n-m} is an even integer, so that $h(2^{n-m}) = h(0) = 0$ by the 2-periodicity of h. Thus

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m}) = \sum_{n=0}^{m} \frac{1}{2^m} = \frac{m+1}{2^m},$$

which gives us

$$\frac{g(x_m) - g(0)}{x_m} = 2^m g(x_m) = m + 1.$$

Since $\lim_{m\to\infty} x_m = 0$ and

$$\lim_{m\to\infty} \frac{g(x_m)}{x_m} = \lim_{m\to\infty} (m+1) = +\infty,$$

it follows that the limit $\lim_{x\to 0} \frac{g(x)}{x}$ does not exist, i.e. $g'(0)$ does not exist.

Exercise 5.4.6.

(a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(1/2)$ does not exist.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = p/2^k$ where $p \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$.

Solution.

(a) These are both special cases of the result in part (b); for the sake of brevity, we omit these proofs.

(b) Let (x_m) be the sequence defined by $x_m = x + 2^{-m} = p2^{-k} + 2^{-m}$. Since we are interested in the limit behaviour as $m \to \infty$, we may assume that $m > k$. Let $n \in \{0, 1, 2, ...\}$ be given. If $n > m > k$ then $p2^{n-k} + 2^{n-m}$ is an even integer and it follows from the 2-periodicity of h that

$$h(2^n x_m) = h\big(p2^{n-k} + 2^{n-m}\big) = h(0) = 0.$$

If $k < n \leq m$ then $p2^{n-k}$ is an even integer and $0 < 2^{n-m} \leq 1$; the 2-periodicity of h implies that

$$h(2^n x_m) = h\big(p2^{n-k} + 2^{n-m}\big) = h(2^{n-m}) = 2^{n-m}.$$

Suppose that $0 \leq n \leq k < m$. Using Euclidean division, we can find integers q and r such that

$$p2^{n-k} = q + r2^{n-k} \quad \text{and} \quad 0 \leq r2^{n-k} \leq \tfrac{1}{2}.$$

Note that $0 < 2^{n-m} \leq \frac{1}{2}$, so that

$$0 < r2^{n-k} + 2^{n-m} \leq 1 \quad \text{and} \quad -1 < -1 + r2^{n-k} + 2^{n-m} \leq 0.$$

If q is even then

$$\begin{aligned} h(2^n x_m) = h\big(p2^{n-k} + 2^{n-m}\big) &= h\big(q + r2^{n-k} + 2^{n-m}\big) = h\big(r2^{n-k} + 2^{n-m}\big) \\ &= r2^{n-k} + 2^{n-m} = h\big(p2^{n-k}\big) + 2^{n-m} = h(2^n x) + 2^{n-m}, \end{aligned}$$

and if q is odd then

$$\begin{aligned} h(2^n x_m) = h\big(p2^{n-k} + 2^{n-m}\big) &= h\big(q + r2^{n-k} + 2^{n-m}\big) = h\big(-1 + r2^{n-k} + 2^{n-m}\big) \\ &= 1 - r2^{n-k} - 2^{n-m} = h\big(p2^{n-k}\big) - 2^{n-m} = h(2^n x) - 2^{n-m}. \end{aligned}$$

In either case we have

$$h(2^n x_m) = h(2^n x) \pm 2^{n-m},$$

with the sign depending on the integer p; this sign will not be important. To summarize:

$$h(2^n x_m) = \begin{cases} h(2^n x) \pm 2^{n-m} & \text{if } 0 \le n \le k < m, \\ 2^{n-m} & \text{if } k < n \le m, \\ 0 & \text{if } n > m > k. \end{cases}$$

Notice that

$$\begin{aligned} g(x) &= g\big(p2^{-k}\big) \\ &= h\big(p2^{-k}\big) + 2^{-1}h\big(p2^{1-k}\big) + \cdots + 2^{-k}h(p) + 2^{-k-1}h(2p) + 2^{-k-2}h(2^2p) + \cdots \\ &= h\big(p2^{-k}\big) + 2^{-1}h\big(p2^{1-k}\big) + \cdots + 2^{-k}h(p) \\ &= \sum_{n=0}^{k} 2^{-n}h\big(p2^{n-k}\big) \\ &= \sum_{n=0}^{k} 2^{-n}h(2^n x). \end{aligned}$$

It follows that

$$\begin{aligned} g(x_m) = \sum_{n=0}^{\infty} 2^{-n}h(2^n x_m) = \sum_{n=0}^{k}(2^{-n}h(2^n x) \pm 2^{-m}) + \sum_{n=k+1}^{m} 2^{-m} \\ = g(x) + (k+1)(\pm 2^{-m}) + (m-k)(2^{-m}). \end{aligned}$$

Thus

$$\frac{g(x_m) - g(x)}{x_m - x} = (k+1)(\pm 1) + m - k = m + K,$$

where $K = (k+1)(\pm 1) - k$ is some integer which depends only on x. Since $\lim_{m\to\infty} x_m = x$ and

$$\lim_{m\to\infty} \frac{g(x_m) - g(x)}{x_m - x} = \lim_{m\to\infty} (m + K) = +\infty,$$

it follows that the limit $\lim_{t\to x} \frac{g(t)-g(x)}{t-x}$ does not exist, i.e. $g'(x)$ does not exist.

Exercise 5.4.7.

(a) First prove the following general lemma: Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, show

$$f'(a) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Now use this lemma to show that $g'(x)$ does not exist.

Solution.

(a) Let us first prove an auxiliary result. Suppose $(x_n), (y_n)$, and (λ_n) are sequences such that $\lim x_n = \lim y_n = x$ and $|\lambda_n| \leq B$ for all $n \in \mathbf{N}$ and some $B \geq 0$. We claim that

$$\lim(\lambda_n x_n + (1 - \lambda_n) y_n) = x.$$

Indeed, observe that

$$\begin{aligned} |\lambda_n x_n + (1-\lambda_n) y_n - x| &= |\lambda_n (x_n - x) + (1 - \lambda_n)(y_n - x)| \\ &\leq |\lambda_n||x_n - x| + |1 - \lambda_n||y_n - x| \\ &\leq (1 + B)(|x_n - x| + |y_n - x|). \end{aligned}$$

Since $(1 + B)(|x_n - x| + |y_n - x|) \to 0$, the Squeeze Theorem proves our claim.

Returning to the exercise, Theorem 4.2.3 implies that

$$\lim_{n\to\infty} \frac{f(a_n) - f(a)}{a_n - a} = \lim_{n\to\infty} \frac{f(b_n) - f(a)}{b_n - a} = f'(a).$$

Note that for each $n \in \mathbf{N}$ we have

$$1 - \frac{a_n - a}{a_n - b_n} = \frac{b_n - a}{b_n - a_n} \quad \text{and} \quad \left| \frac{a_n - a}{a_n - b_n} \right| < 1.$$

Furthermore,

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{a_n - a}{a_n - b_n} \frac{f(a_n) - f(a)}{a_n - a} + \frac{b_n - a}{b_n - a_n} \frac{f(b_n) - f(a)}{b_n - a}$$

for each $n \in \mathbf{N}$. Taking

$$x_n = \frac{f(a_n) - f(a)}{a_n - a}, \quad y_n = \frac{f(b_n) - f(a)}{b_n - a}, \quad \text{and} \quad \lambda_n = \frac{a_n - a}{a_n - b_n}$$

in our auxiliary result shows that

$$f'(a) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Recall that for each $n \in \{0, 1, 2, ...\}$ the function $h_n : \mathbf{R} \to \mathbf{R}$ is given by $h_n(x) = h(2^n x)$. Each h_n is a piecewise linear function which has corners, i.e. fails to be differentiable, at each dyadic rational of the form $a2^{-n}$. Note that h_n is linear on each interval of the form $[a2^{-n}, (a + 1)2^{-n}]$; in particular, h_n is differentiable on $(a2^{-n}, (a + 1)2^{-n})$, with slope given by ± 1. Recall also that for each $m \in \{0, 1, 2, ...\}$, the function $g_m : \mathbf{R} \to \mathbf{R}$ is defined as

$$g_m(x) = \sum_{n=0}^{m} 2^{-n} h_n(x) = \sum_{n=0}^{m} 2^{-n} h(2^n x).$$

Each g_m is a linear combination of piecewise linear functions and hence is itself a piecewise linear function. Consider two adjacent dyadic rationals $p2^{-m}$ and $(p + 1)2^{-m}$. By our previous discussion, for each $0 \leq n \leq m$, the function h_n is linear on

$[p2^{-m}, (p+1)2^{-m}]$ and hence differentiable on $(p2^{-m}, (p+1)2^{-m})$. It follows that g_m is linear on $[p2^{-m}, (p+1)2^{-m}]$ and hence differentiable on $(p2^{-m}, (p+1)2^{-m})$, with slope given by

$$g_m'(x) = \frac{g_m((p+1)2^{-m}) - g_m(p2^{-m})}{2^{-m}}$$

for $x \in (p2^{-m}, (p+1)2^{-m})$.

Let x, (x_m), and (y_m) be defined as in the textbook. Given the previous discussion, for each $m \in \{0, 1, 2, ...\}$ we have

$$g_m'(x) = \frac{g_m(y_m) - g_m(x_m)}{y_m - x_m}.$$

In fact, since $h_n(x_m) = h_n(y_m) = 0$ for all $n > m$, we actually have $g(y_m) = g_m(y_m)$ and $g(x_m) = g_m(x_m)$, so that

$$\frac{g(y_m) - g(x_m)}{y_m - x_m} = \frac{g_m(y_m) - g_m(x_m)}{y_m - x_m} = g_m'(x).$$

Now observe that

$$g_{m+1}(t) - g_m(t) = 2^{-m-1} h_{m+1}(t).$$

As we noted earlier, each of the functions g_{m+1}, g_m, and h_{m+1} is differentiable at x since x is not a dyadic rational. It follows from the usual rules of differentiation that

$$\left|g_{m+1}'(x) - g_m'(x)\right| = \left|h_{m+1}'(x)\right| = |\pm 1| = 1.$$

This implies that the sequence $(g_m'(x))_{m=0}^{\infty}$ is not convergent, i.e. the sequence

$$\frac{g(y_m) - g(x_m)}{y_m - x_m}$$

does not converge. By the contrapositive of the result proved in part (a), we see that g is not differentiable at x.

Exercise 5.4.8. Review the argument for the nondifferentiability of $g(x)$ at nondyadic points. Does the argument still work if we replace $g(x)$ with the summation $\sum_{n=0}^{\infty}(1/2^n)h(3^n x)$? Does the argument work for the function $\sum_{n=0}^{\infty}(1/3^n)h(2^n x)$?

Solution. Let $g(x) = \sum_{n=0}^{\infty} 2^{-n} h(3^n x)$ and $g_m(x) = \sum_{n=0}^{m} 2^{-n} h(3^n x)$. The argument from Exercise 5.4.7 (b) should be repeated considering 3-adic rational numbers, i.e. rationals of the form $p3^{-k}$ for some $p \in \mathbf{Z}$ and some $k \in \{0, 1, 2, ...\}$. The argument still works, with one small difference. If x is not a 3-adic rational number then similar reasoning shows that g_m is differentiable at x. The difference this time is that

$$\left|g_{m+1}'(x) - g_m'(x)\right| = \left(\tfrac{3}{2}\right)^{m+1}.$$

Since this does not converge to zero, we see that the sequence $(g'_m(x))_{m=0}^{\infty}$ is not convergent and we may conclude that $g'(x)$ does not exist.

Now let $g(x) = \sum_{n=0}^{\infty} 3^{-n} h(2^n x)$ and $g_m(x) = \sum_{n=0}^{m} 3^{-n} h(2^n x)$. We again consider dyadic rationals and arrive at

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \left(\tfrac{2}{3}\right)^{m+1}$$

for an x which is not a dyadic rational number. Since this does converge to zero, our argument breaks down here. In fact, Theorem 6.4.3 shows that g is differentiable at every such x.

Chapter 6. Sequences and Series of Functions

6.2. Uniform Convergence of a Sequence of Functions

Exercise 6.2.1. Let

$$f_n(x) = \frac{nx}{1 + nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

(b) Is the convergence uniform on $(0, \infty)$?

(c) Is the convergence uniform on $(0, 1)$?

(d) Is the convergence uniform on $(1, \infty)$?

Solution.

(a) Fix $x \in (0, \infty)$ and observe that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1 + nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x^2} = \frac{1}{x}.$$

Thus, letting $f : (0, \infty) \to \mathbf{R}$ be the function given by $f(x) = \frac{1}{x}$, we see that (f_n) converges pointwise to f.

(b) The convergence is not uniform on $(0, \infty)$. To argue this, let us negate the definition of uniform convergence. A sequence of functions $(f_n : A \to \mathbf{R})$ does not converge uniformly to a function $f : A \to \mathbf{R}$ if there exists an $\varepsilon > 0$ such that for all $N \in \mathbf{N}$, we can find an $x \in A$ and an $n \geq N$ such that $|f_n(x) - f(x)| \geq \varepsilon$. In symbols:

$$(\exists \varepsilon > 0)(\forall N \in \mathbf{N})(\exists x \in A)(\exists n \geq N)(|f_n(x) - f(x)| \geq \varepsilon).$$

Let $N \in \mathbf{N}$ be given and observe that

$$\frac{1}{N+1} \in (0,1) \subseteq (0, \infty) \quad \text{and} \quad \left|f_{N+1}\left(\tfrac{1}{N+1}\right) - f\left(\tfrac{1}{N+1}\right)\right| = \frac{(N+1)^2}{N+2} \geq \frac{4}{3}.$$

Thus the convergence $f_n \to f$ is not uniform on $(0, \infty)$.

(c) The convergence is not uniform on $(0, 1)$, as part (a) shows.

(d) The convergence is uniform on $(1, \infty)$. Let $\varepsilon > 0$ be given and let $N \in \mathbf{N}$ be such that $N > \frac{1}{\varepsilon} - 1$. For all $x \in (1, \infty)$ and all $n \geq N$ it follows that

$$|f_n(x) - f(x)| = \frac{1}{x(1 + nx^2)} \leq \frac{1}{1+n} \leq \frac{1}{1+N} < \varepsilon.$$

Exercise 6.2.2.

(a) Define a sequence of functions on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n.

Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbf{R}$? Is f continuous at zero?

(b) Repeat this exercise using the sequence of functions

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem (Theorem 6.2.6).

Solution.

(a) Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $f_n \to f$ pointwise. If $x \notin \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ then

$$|f_n(x) - f(x)| = |0 - 0| = 0,$$

and if $x = \frac{1}{N}$ for some $N \in \mathbf{N}$ then for all $n \geq N$ we have

$$|f_n(x) - f(x)| = |1 - 1| = 0.$$

Each f_n is continuous at zero since each f_n is identically zero on the interval $\left(-\infty, \frac{1}{n}\right)$, however f is not continuous at zero since $\frac{1}{n} \to 0$ and $f\left(\frac{1}{n}\right) \to 1 \neq 0 = f(0)$. It follows that the convergence $f_n \to f$ cannot be uniform. otherwise the Continuous Limit Theorem (Theorem 6.2.6) would be violated.

(b) Define $g : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $g_n \to g$ pointwise. If $x \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ then

$$|g_n(x) - g(x)| = |0 - 0| = 0,$$

and if $x = \frac{1}{N}$ for some $N \in \mathbf{N}$ then for all $n \geq N$ we have

$$|g_n(x) - g(x)| = |x - x| = 0.$$

Each g_n is continuous at zero since each g_n is identically zero on the interval $\left(-\infty, \frac{1}{n}\right)$. The convergence $g_n \to g$ is uniform since for any $n \in \mathbf{N}$ and $x \in \mathbf{R}$ we have

$$|g_n(x) - g(x)| \leq \frac{1}{n+1}.$$

The Continuous Limit Theorem (Theorem 6.2.6) implies that g must be continuous at zero, and this is straightforward to verify directly.

(c) Define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, ..., \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $h_n \to h$ pointwise. If $x \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ then

$$|h_n(x) - h(x)| = |0 - 0| = 0,$$

and if $x = \frac{1}{N}$ for some $N \in \mathbf{N}$ then for all $n \geq N+1$ we have

$$|h_n(x) - h(x)| = |x - x| = 0.$$

Each h_n is continuous at zero since each h_n is identically zero on the interval $\left(-\infty, \frac{1}{n}\right)$. The convergence here is not uniform: for any $N \in \mathbf{N}$ observe that

$$\left|h_{N+1}\left(\tfrac{1}{N+1}\right) - h\left(\tfrac{1}{N+1}\right)\right| = 1 - \frac{1}{N+1} \geq \frac{1}{2}.$$

However, h is continuous at zero. This does not contradict the Continuous Limit Theorem (Theorem 6.2.6), but it does show that the converse does not hold.

Exercise 6.2.3. For each $n \in \mathbf{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1 + x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n. \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

(a) Find the pointwise limit on $[0, \infty)$.

(b) Explain how we know the convergence cannot be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution.

(a) Let $g : [0, \infty) \to \mathbf{R}$ be given by

$$g(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

We claim that $g_n \to g$ pointwise. Observe that if $0 \le x < 1$ then $\lim_{n\to\infty} x^n = 0$ and thus $\lim_{n\to\infty} g_n(x) = x$; if $x = 1$ then $g_n(x) = \frac{1}{2}$ for all $n \in \mathbf{N}$; and if $x > 1$ then $\lim_{n\to\infty} x^n = +\infty$ and thus $\lim_{n\to\infty} g_n(x) = 0$. Our claim follows.

Let $h : [0, \infty) \to \mathbf{R}$ be the function given by

$$h(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that $h_n \to h$ pointwise. Indeed, $h_n(0) = h(0) = 0$ for all $n \in \mathbf{N}$, and if $x > 0$ then choose $N \in \mathbf{N}$ such that $\frac{1}{N} \le x$ and observe that

$$n \ge N \quad \Rightarrow \quad |h_n(x) - h(x)| = |1 - 1| = 0.$$

(b) The convergence cannot be uniform on $[0, \infty)$ by the Continuous Limit Theorem (Theorem 6.2.6): each f_n and each g_n is continuous on $[0, \infty)$ but neither f nor g is continuous on $[0, \infty)$.

(c) Restrict each g_n and g to the domain $[2, \infty)$, so that g is the constant function $g(x) = 0$. We claim that $g_n \to g$ uniformly on $[2, \infty)$. We will make use of the inequality

$$x^n + 1 > x^n - 1 = (1 + x + \cdots + x^{n-1})(x - 1) \ge n(x - 1),$$

which holds for all $n \in \mathbf{N}$ and $x \ge 2$. Observe that

$$|g_n(x) - g(x)| = g_n(x) = \frac{x}{x^n + 1} \le \frac{x}{n(x-1)} \le \frac{2}{n},$$

where we have used that $\frac{x}{x-1} = 1 + \frac{1}{x-1} \le 2$ for all $x \ge 2$. The uniform convergence follows since the bound $\frac{2}{n}$ converges to zero and does not depend on x.

Restrict each h_n and h to the domain $[1, \infty)$, so that $h_n(x) = h(x) = 1$ for all $n \in \mathbf{N}$ and $x \ge 1$; certainly the convergence $h_n \to h$ is uniform.

Exercise 6.2.4. Review Exercise 5.2.8 which includes the definition for a uniformly differentiable function. Use the results discussed in Section 6.2 to show that if f is uniformly differentiable, then f' is continuous.

Solution. Suppose $f : A \to \mathbf{R}$ is uniformly differentiable, where A is some open interval. For each $n \in \mathbf{N}$ define $f_n : A \to \mathbf{R}$ by

$$f_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{n^{-1}}.$$

For any $x \in A$ the existence of $f'(x)$ implies that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f\left(x+\frac{1}{n}\right) - f(x)}{n^{-1}} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Thus $f_n \to f'$ pointwise on A. We claim that this convergence is uniform. Let $\varepsilon > 0$ be given. Because f is uniformly differentiable there exists a $\delta > 0$ such that

$$x, y \in A \text{ and } 0 < |x-y| < \delta \quad \Rightarrow \quad \left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < \varepsilon.$$

Let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \delta$. For any $n \geq N$ and $x \in A$ note that $\left|x + \frac{1}{n} - x\right| = \frac{1}{n} < \delta$; it follows that

$$|f_n(x) - f(x)| = \left|\frac{f\left(x+\frac{1}{n}\right) - f(x)}{n^{-1}} - f'(x)\right| < \varepsilon.$$

Thus $f_n \to f'$ uniformly on A. Since each f_n is continuous on A the Continuous Limit Theorem (Theorem 6.2.6) allows us to conclude that f' is continuous.

Exercise 6.2.5. Using the Cauchy Criterion for convergent sequences of real numbers (Theorem 2.6.4), supply a proof for Theorem 6.2.5. (First, define a candidate for $f(x)$, and then argue that $f_n \to f$ uniformly.)

Solution. Let $(f_n : A \to \mathbf{R})$ be a sequence of functions. Suppose that (f_n) converges uniformly to a function $f : A \to \mathbf{R}$ and let $\varepsilon > 0$ be given. By uniform convergence, there is an $N \in \mathbf{N}$ such that

$$x \in A \text{ and } n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

It follows that for any $x \in A$ and any $n, m \geq N$ we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now suppose that for any $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that

$$x \in A \text{ and } n, m \geq N \quad \Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon. \tag{1}$$

Note that, for any given $x \in A$, this implies that the sequence of real numbers $(f_n(x))$ is a Cauchy sequence. The completeness of $\mathbf{R}$ then implies that $\lim_{n\to\infty} f_n(x)$ exists. Define $f : A \to \mathbf{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$, so that $f_n \to f$ pointwise. Our claim is that this convergence is uniform. Let $\varepsilon > 0$ be given. By assumption there exists an $N \in \mathbf{N}$ such that (1) holds. Temporarily fix $x \in A$ and $n \geq N$ and observe that for every $m \geq N$ we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + |f_m(x) - f(x)|.$$

The Order Limit Theorem (Theorem 2.3.4) applied to the inequality above, treating both sides as sequences of m, implies that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} + \lim_{m\to\infty} |f_m(x) - f(x)| = \frac{\varepsilon}{2} < \varepsilon.$$

It follows that

$$x \in A \text{ and } n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

Thus $f_n \to f$ uniformly on A, as claimed.

Exercise 6.2.6. Assume $f_n \to f$ on a set A. Theorem 6.2.6 is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that *all* of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each f_n is uniformly continuous, then f is uniformly continuous.

(b) If each f_n is bounded, then f is bounded.

(c) If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

(d) If each f_n has fewer than M discontinuities (where $M \in \mathbf{N}$ is fixed), then f has fewer than M discontinuities.

(e) If each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

Solution.

(a) Let $(f_n : [0,1] \to \mathbf{R})$ be the sequence of functions defined by $f_n(x) = x^n$ and let $f : [0,1] \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

As shown in Example 6.2.2 (ii), $f_n \to f$ pointwise. Each f_n is a continuous function defined on the compact domain $[0,1]$ and thus each f_n is uniformly continuous by Theorem 4.4.7. However, f is not continuous and hence not uniformly continuous.

We claim that uniform convergence preserves uniform continuity. Suppose that $(f_n : A \to \mathbf{R})$ is a sequence of uniformly continuous functions which converges uniformly to a function $f : A \to \mathbf{R}$. Let $\varepsilon > 0$ be given. By uniform convergence, there is an $N \in \mathbf{N}$ such that

$$x \in A \text{ and } n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

The function f_N is uniformly continuous by assumption and thus there exists a $\delta > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta \quad \Rightarrow \quad |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}.$$

Now suppose that $x, y \in A$ are such that $|x - y| < \delta$ and observe that

$$|f(x) - f(y)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus f is uniformly continuous.

(b) Let $(f_n : (0, \infty) \to \mathbf{R})$ be the sequence of functions defined by

$$f_n(x) = \frac{nx}{1 + nx^2}$$

and let $f : (0, \infty) \to \mathbf{R}$ be the function defined by $f(x) = \frac{1}{x}$. As we showed in Exercise 6.2.1, $f_n \to f$ pointwise. For any given $n \in \mathbf{N}$ we have

$$f_n(x) \le nx \le n \text{ on } (0, 1] \quad \text{and} \quad f_n(x) \le 1 \text{ on } (1, \infty).$$

Thus each f_n is bounded, whereas f is unbounded.

We claim that uniform convergence preserves boundedness. Suppose that $(f_n : A \to \mathbf{R})$ is a sequence of bounded functions (the bound may depend on n) which converges uniformly to a function $f : A \to \mathbf{R}$. By uniform convergence there is an $N \in \mathbf{N}$ such that

$$|f_N(x) - f(x)| < 1 \text{ for all } x \in A.$$

By assumption the function f_N is bounded, i.e. there is an $M > 0$ such that $|f_N(x)| \le M$ for all $x \in A$. It follows that

$$|f(x)| \le |f_N(x)| + |f_N(x) - f(x)| < M + 1$$

for every $x \in A$. Thus f is bounded.

(c) Let $(f_n : \mathbf{R} \to \mathbf{R})$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

As we showed in Exercise 6.2.2 (b), $f_n \to f$ pointwise. For a given $n \in \mathbf{N}$ the function f_n is discontinuous precisely on the finite set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$, whereas f is discontinuous precisely on the infinite set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. As shown in Exercise 6.2.2 (b), the convergence here is uniform, demonstrating that uniform convergence need not preserve the finiteness of the set of discontinuities.

(d) If we define (f_n) and f as in part (a), then each f_n has zero discontinuities but f has a discontinuity at $x = 1$.

The proposition is true if we assume uniform convergence. To see this, let us prove the following lemma.

Lemma L.14. Suppose $(f_n : A \to \mathbf{R})$ is a sequence of functions converging uniformly to a function $f : A \to \mathbf{R}$. If f is discontinuous at $c \in \mathbf{R}$ then there exists an $N \in \mathbf{N}$ such that f_n is discontinuous at $c \in \mathbf{R}$ for all $n \geq N$.

Proof. Since f is discontinuous at c there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there is an $x_\delta \in A$ satisfying

$$|x_\delta - c| < \delta \quad \text{and} \quad |f(x_\delta) - f(c)| \geq \varepsilon. \tag{1}$$

By uniform convergence there is an $N \in \mathbf{N}$ such that

$$x \in A \quad \text{and} \quad n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \tfrac{\varepsilon}{4}.$$

Let $\delta > 0$ be given, so that there exists an $x_\delta \in A$ such that (1) holds. Suppose $n \geq N$ and observe that

$$\begin{aligned} \varepsilon \leq |f(x_\delta) - f(c)| &\leq |f_n(x_\delta) - f_n(c)| + |f_n(x_\delta) - f(x_\delta)| + |f_n(c) - f(c)| \\ &< |f_n(x_\delta) - f_n(c)| + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = |f_n(x_\delta) - f_n(c)| + \tfrac{\varepsilon}{2}. \end{aligned}$$

Thus $|f_n(x_\delta) - f_n(c)| > \frac{\varepsilon}{2}$. It follows that f_n is discontinuous at c for all $n \geq N$. □

We can now prove the proposition, assuming uniform convergence, by proving the contrapositive. Suppose $(f_n : A \to \mathbf{R})$ is a sequence of functions converging uniformly to a function $f : A \to \mathbf{R}$ and suppose that f has at least M discontinuities, say $x_1, \ldots, x_M$. By Lemma L.14 there exist positive integers $N_1, \ldots, N_M$ such that

$$n \geq N_m \quad \Rightarrow \quad f_n \text{ is discontinuous at } x_m$$

for each $m \in \{1, \ldots, M\}$. If we let $N = \max\{N_1, \ldots, N_M\}$ then f_N is discontinuous at each point $x_1, \ldots, x_M$ and thus f_N has at least M discontinuities.

(e) Let $(f_n : \mathbf{R} \to \mathbf{R})$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{a}{b} \text{ with } b \leq n, \\ 0 & \text{otherwise} \end{cases}$$

(where we assume $a \in \mathbf{Z}, b \in \mathbf{N}$, and $\gcd(a, b) = 1$), and let $f : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function, i.e.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Notice that $f_n \to f$ pointwise: if $x \notin \mathbf{Q}$ then $f_n(x) = f(x) = 0$ for every $n \in \mathbf{N}$, and if $x = \frac{a}{b} \in \mathbf{Q}$ then $f_n(x) = f(x) = 1$ for all $n \geq b$. Notice further that for a given $n \in \mathbf{N}$ the function f_n is discontinuous precisely on the countable set

$$\bigcup_{b=1}^{n} \left\{ \frac{a}{b} : a \in \mathbf{Z} \right\},$$

whereas f is discontinuous on the uncountable set $\mathbf{R}$.

The proposition is true if we assume uniform convergence. Let

$$D_f = \{x \in \mathbf{R} : f \text{ is discontinuous at } x\}.$$

It follows from Lemma L.14 that

$$D_f \subseteq \bigcup_{n=1}^{\infty} D_{f_n}.$$

By assumption each D_{f_n} is at most countable and thus the union $\bigcup_{n=1}^{\infty} D_{f_n}$ is at most countable by Theorem 1.5.8 (ii). It follows that D_f is at most countable.

Exercise 6.2.7. Let f be uniformly continuous on all of $\mathbf{R}$, and define a sequence of functions by $f_n(x) = f\left(x + \frac{1}{n}\right)$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbf{R}$.

Solution. Let $\varepsilon > 0$ be given. By the uniform continuity of f there exists a $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \delta$. For any $n \geq N$ and $x \in \mathbf{R}$ we have $\left|x + \frac{1}{n} - x\right| = \frac{1}{n} < \delta$ and thus

$$|f_n(x) - f(x)| = \left|f\left(x + \tfrac{1}{n}\right) - f(x)\right| < \varepsilon.$$

Hence $f_n \to f$ uniformly.

For a counterexample to the proposition assuming only continuity, consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$. Theorem 4.4.5 with the sequences $x_n = n + \frac{1}{n}$ and $y_n = n$ shows that f is not uniformly continuous on $\mathbf{R}$. Furthermore, for any $N \in \mathbf{N}$ observe that

$$|f_N(N) - f(N)| = 2 + \frac{1}{N^2} > 2.$$

It follows that (f_n) does not converge uniformly to f.

Exercise 6.2.8. Let (g_n) be a sequence of continuous functions that converges uniformly to g on a compact set K. If $g(x) \neq 0$ on K, show $(1/g_n)$ converges uniformly on K to $1/g$.

Solution. First let us show that the sequence (g_n) is eventually non-zero, so that the sequence of reciprocals $(1/g_n)$ is eventually well-defined. Note g must be continuous on K by the Continuous Limit Theorem (Theorem 6.2.6). It follows that $|g|$ is continuous on the compact set K and hence attains a minimum by the Extreme Value Theorem, say $0 < M \leq |g(x)|$ for all $x \in K$; note that M must be strictly positive since $g \neq 0$ on K. By uniform convergence there is an $N_1 \in \mathbf{N}$ such that

$$x \in K \text{ and } n \geq N_1 \quad \Rightarrow \quad |g_n(x) - g(x)| < \frac{M}{2} \quad \Rightarrow \quad 0 < \frac{M}{2} < |g_n(x)| \quad \Rightarrow \quad g_n(x) \neq 0.$$

Thus the sequence $(1/g_n)$ is well-defined for all $n \geq N_1$.

To show that $(1/g_n)$ converges uniformly to $1/g$, observe that by uniform convergence there is an $N_2 \in \mathbf{N}$ such that

$$x \in K \text{ and } n \geq N_2 \quad \Rightarrow \quad |g_n(x) - g(x)| < \frac{M^2}{2}\varepsilon.$$

Let $N = \max\{N_1, N_2\}$ and suppose $x \in K$ and $n \geq N$. It follows that

$$\left|\frac{1}{g_n(x)} - \frac{1}{g(x)}\right| = \left|\frac{g_n(x) - g(x)}{g_n(x)g(x)}\right| \leq \frac{2|g_n(x) - g(x)|}{M^2} < \varepsilon.$$

Thus $1/g_n \to 1/g$ uniformly.

Exercise 6.2.9. Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbf{N}$, then $(f_n g_n)$ does converge uniformly.

Solution. Suppose that each f_n and each g_n is defined on some domain $A \subseteq \mathbf{R}$ and suppose that $f_n \to f$ uniformly and $g_n \to g$ uniformly for some functions $f, g : A \to \mathbf{R}$.

(a) We claim that $(f_n + g_n)$ converges uniformly to $f + g$. Let $\varepsilon > 0$ be given. By the uniform convergence of $f_n \to f$ and $g_n \to g$, there exist positive integers N_1, N_2 such that

$$x \in A \text{ and } n \geq N_1 \quad \Rightarrow \quad |f_n(x) - f(x)| < \tfrac{\varepsilon}{2},$$

$$x \in A \text{ and } n \geq N_2 \quad \Rightarrow \quad |g_n(x) - g(x)| < \tfrac{\varepsilon}{2}.$$

It follows that for any $x \in A$ and any $n \geq \max\{N_1, N_2\}$ we have

$$|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Thus $f_n + g_n \to f + g$ uniformly.

(b) Let $(f_n : \mathbf{R} \to \mathbf{R})$ be the sequence defined by $f_n(x) = x + \frac{1}{n}$ and let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x$. It is straightforward to argue that $f_n \to f$ uniformly. Observe that $f_n^2 : \mathbf{R} \to \mathbf{R}$ and $f^2 : \mathbf{R} \to \mathbf{R}$ are given by

$$[f_n(x)]^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \quad \text{and} \quad [f(x)]^2 = x^2.$$

It follows that $f_n^2 \to f^2$ pointwise, but the convergence is not uniform: for any $N \in \mathbf{N}$ we have

$$\left|[f_N(N)]^2 - [f(N)]^2\right| = 2 + \frac{1}{N^2} > 2.$$

(c) Since each f_n is bounded, Exercise 6.2.6 (b) shows that f is bounded, say $|f(x)| \leq L$ for all $x \in A$ and some $L > 0$. Let $\varepsilon > 0$ be given. By the uniform convergence of $f_n \to f$ and $g_n \to g$, there exist positive integers N_1, N_2 such that

$$x \in A \text{ and } n \geq N_1 \quad \Rightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2M},$$
$$x \in A \text{ and } n \geq N_2 \quad \Rightarrow \quad |g_n(x) - g(x)| < \frac{\varepsilon}{2L}.$$

It follows that for any $x \in A$ and any $n \geq \max\{N_1, N_2\}$ we have

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &\leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)| \\ &\leq M|f_n(x) - f(x)| + L|g_n(x) - g(x)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus $f_n g_n \to fg$ uniformly.

Exercise 6.2.10. This exercise and the next explore partial converses of the Continuous Limit Theorem (Theorem 6.2.6). Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Solution. First, let us prove a couple of useful lemmas.

Lemma L.15. If $(f_n : A \to \mathbf{R})$ is a sequence of increasing functions converging pointwise to a function $f : A \to \mathbf{R}$ then f is increasing.

Proof. Let $x \leq y$ in A be given. By assumption we have $f_n(x) \leq f_n(y)$ for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) and the pointwise convergence $f_n \to f$ then imply that $f(x) \leq f(y)$. □

Lemma L.16. If $f, g : [c, d] \to \mathbf{R}$ are increasing functions then for all $x \in [c, d]$ the following inequality holds:

$$|f(x) - g(x)| \leq \max\{|f(c) - g(d)|, |f(d) - g(c)|\}.$$

Proof. Let $x \in [c, d]$ be given. Since f and g are increasing we have

$$f(c) \leq f(x) \leq f(d) \quad \text{and} \quad g(c) \leq g(x) \leq g(d).$$

Together these imply that

$$f(x) - g(x) \leq f(d) - g(c) \leq |f(d) - g(c)| \leq \max\{|f(c) - g(d)|, |f(d) - g(c)|\},$$
$$g(x) - f(x) \leq g(d) - f(c) \leq |f(c) - g(d)| \leq \max\{|f(c) - g(d)|, |f(d) - g(c)|\}.$$

The desired inequality follows. □

Returning to the exercise, let $\varepsilon > 0$ be given. Because f is continuous on the compact set $[a, b]$, it must be uniformly continuous here (Theorem 4.4.7). Consequently, there exists a $\delta > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \tfrac{\varepsilon}{2}. \tag{1}$$

Choose $K \in \mathbf{N}$ such that $K^{-1}(b - a) < \delta$ and for $i \in \{0, ..., K\}$ let $x_i = a + iK^{-1}(b - a)$, so that

$$x_0 = a, \quad x_K = b, \quad \text{and} \quad x_{i+1} - x_i = \frac{b - a}{K} < \delta.$$

This partitions the interval $[a, b]$ into subintervals $[x_i, x_{i+1}]$ of equal length, such that this length is less than δ. The pointwise convergence $f_n \to f$ implies that for each $i \in \{0, ..., K\}$ there is an $N_i \in \mathbf{N}$ such that

$$n \geq N_i \quad \Rightarrow \quad |f_n(x_i) - f(x_i)| < \tfrac{\varepsilon}{2}. \tag{2}$$

Let $N = \max\{N_0, ..., N_K\}$ and suppose that $n \geq N$. Fix $x \in [a, b]$ and note that $x \in [x_i, x_{i+1}]$ for some $i \in \{0, ..., K - 1\}$. It follows from (1) and (2) that

$$|f_n(x_{i+1}) - f(x_i)| \leq |f_n(x_{i+1}) - f(x_{i+1})| + |f(x_{i+1}) - f(x_i)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

We can similarly show that $|f_n(x_i) - f(x_{i+1})| < \varepsilon$. Observe that f is increasing by Lemma L.15; it follows from Lemma L.16 that

$$|f_n(x) - f(x)| \leq \max\{|f_n(x_{i+1}) - f(x_i)|, |f_n(x_i) - f(x_{i+1})|\} < \varepsilon.$$

We have now shown that if $x \in [a, b]$ and $n \geq N$ then $|f_n(x) - f(x)| < \varepsilon$. Thus $f_n \to f$ uniformly on $[a, b]$.

Exercise 6.2.11 (Dini's Theorem). Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

(b) Let $\varepsilon > 0$ be arbitrary, and define $K_n = \{x \in K : g_n(x) \geq \varepsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$, and use this observation to finish the argument.

Solution.

(a) The sequence (g_n) converges to zero pointwise on K; for a given $x \in K$, the sequence of real numbers $(g_n(x))$ is decreasing; and each g_n is a continuous function. To complete the proof, it will suffice to show that $g_n \to 0$ uniformly on K.

(b) Suppose $x \in K_{n+1}$ for some $n \in \mathbf{N}$, so that $g_{n+1}(x) \geq \varepsilon$. Since the sequence $(g_n(x))$ is decreasing we then have $g_n(x) \geq g_{n+1}(x) \geq \varepsilon$ and thus $x \in K_n$. It follows that $K_{n+1} \subseteq K_n$ and hence that $\cdots \subseteq K_3 \subseteq K_2 \subseteq K_1$.

If each K_n were non-empty then Theorem 3.3.5 would imply the existence of an $x \in K$ such that $g_n(x) \geq \varepsilon > 0$ for each $n \in \mathbf{N}$, so that $\lim_{n\to\infty} g_n(x) \neq 0$. Taking the contrapositive of this and using our assumption that $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in K$, we see that there exists an $N \in \mathbf{N}$ such that $K_N = \emptyset$, which forces $K_n = \emptyset$ for all $n \geq N$. In other words,

$$x \in K \text{ and } n \geq N \quad \Rightarrow \quad g_n(x) < \varepsilon.$$

Since $g_n \to 0$ pointwise on K and the sequence $(g_n(x))$ is decreasing for any $x \in K$, it must be the case that each g_n is non-negative, so that $|g_n| = g_n$. We may conclude that $g_n \to 0$ uniformly on K.

Exercise 6.2.12 (Cantor Function). Review the construction of the Cantor set $C \subseteq [0, 1]$ from Section 3.1. This exercise makes use of results and notation from this discussion.

(a) Define $f_0(x) = x$ for all $x \in [0, 1]$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \leq x < 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

Sketch f_0 and f_1 over $[0, 1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0, 1] \setminus C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

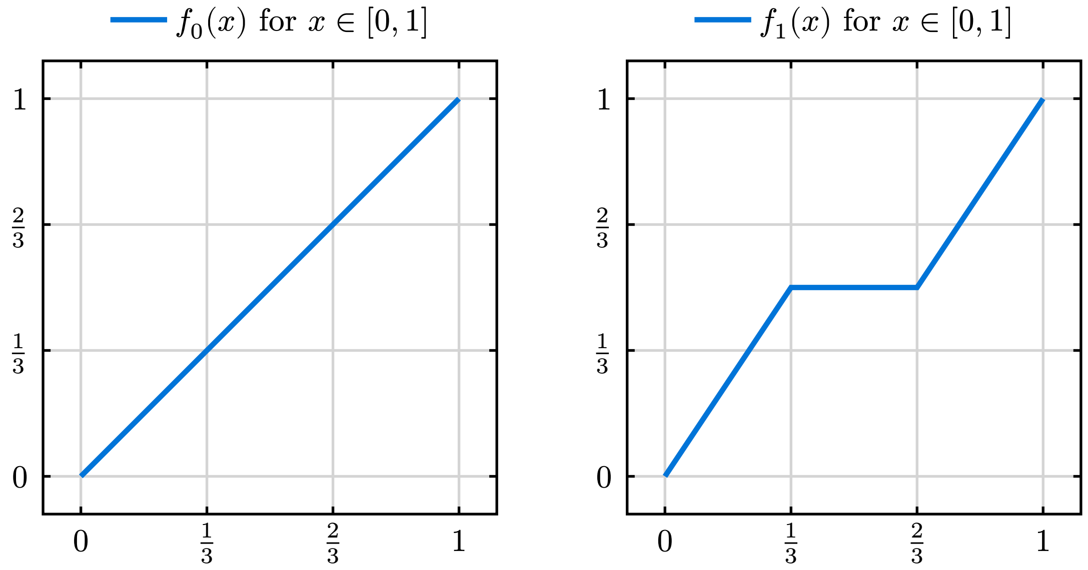
$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x < 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x - 2) + 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

If we continue this process, show that the resulting sequence (f_n) converges uniformly on $[0, 1]$.

(c) Let $f = \lim f_n$. Prove that f is a continuous, increasing function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0, 1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

Solution.

(a) Here are graphs of f_0 and f_1 over $[0, 1]$.

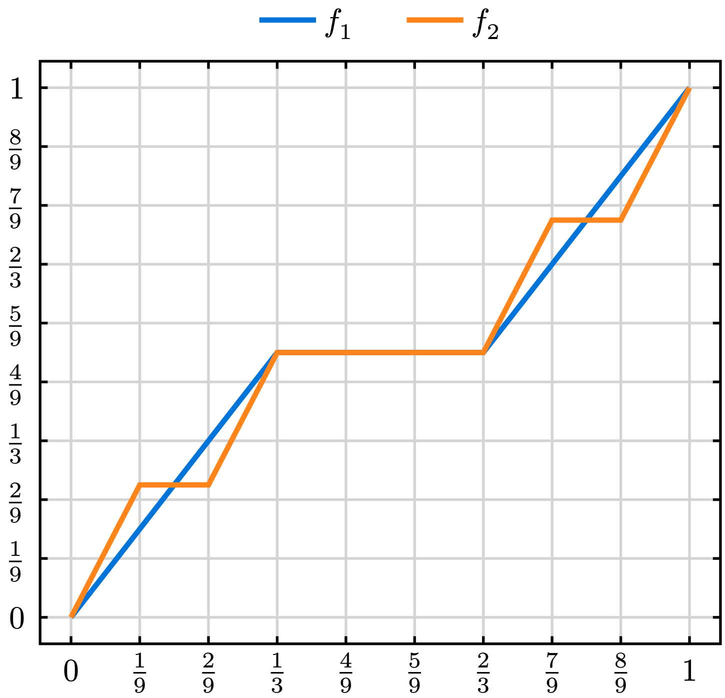


(b) The sequence (f_n) is defined by

$$f_n(x) = \begin{cases} \frac{1}{2} f_{n-1}(3x) & \text{if } 0 \le x \le \frac{1}{3}, \\ f_{n-1}(x) & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} f_{n-1}(3x-2) + \frac{1}{2} & \text{if } \frac{2}{3} \le x \le 1 \end{cases}$$

for $n \ge 2$.

We will show by induction that $\left|f_{n+1}(x) - f_n(x)\right| \le \frac{1}{6} \cdot 2^{-n}$ for all $x \in [0, 1]$ and all $n \in \mathbf{N}$. For the base case $n = 1$, it is straightforward to verify that the maximum of $|f_2(x) - f_1(x)|$ is $\frac{1}{12}$, which is achieved at $x = \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$; see the following graph.



Suppose that $|f_{n+1}(x) - f_n(x)| \le \frac{1}{6} \cdot 2^{-n}$ for all $x \in [0,1]$ some $n \in \mathbf{N}$. There are three cases.

Case 1. For $0 \le x \le \frac{1}{3}$ we have $f_{n+2}(x) = \frac{1}{2} f_{n+1}(3x)$ and $f_{n+1}(x) = \frac{1}{2} f_n(3x)$. It follows that

$$|f_{n+2}(x) - f_{n+1}(x)| = \left|\tfrac{1}{2} f_{n+1}(3x) - \tfrac{1}{2} f_n(3x)\right| = \tfrac{1}{2}|f_{n+1}(3x) - f_n(3x)| \le \tfrac{1}{6} \cdot 2^{-(n+1)},$$

where we have used the induction hypothesis for the last inequality.

Case 2. For $\frac{1}{3} \le x \le \frac{2}{3}$ we have $f_{n+2}(x) = f_{n+1}(x)$ and thus $|f_{n+2}(x) - f_{n+1}(x)| = 0$.

Case 3. For $\frac{2}{3} \le x \le 1$ we have

$$f_{n+2}(x) = \tfrac{1}{2} f_{n+1}(3x-2) + \tfrac{1}{2} \quad \text{and} \quad f_{n+1}(x) = \tfrac{1}{2} f_n(3x-2) + \tfrac{1}{2}.$$

It follows that

$$\begin{aligned} |f_{n+2}(x) - f_{n+1}(x)| &= \left|\tfrac{1}{2} f_{n+1}(3x-2) - \tfrac{1}{2} f_n(3x-2)\right| \\ &= \tfrac{1}{2}|f_{n+1}(3x-2) - f_n(3x-2)| \le \tfrac{1}{6} \cdot 2^{-(n+1)}, \end{aligned}$$

where we have used the induction hypothesis for the last inequality.

This completes the induction step and thus $|f_{n+1}(x) - f_n(x)| \le \frac{1}{6} \cdot 2^{-n}$ for each $n \in \mathbf{N}$. This inequality implies that for any $x \in [0,1]$ and any positive integers $n > m$ we have

$$|f_n(x) - f_m(x)| \le \sum_{j=m}^{n-1} |f_{j+1}(x) - f_j(x)| \le \frac{1}{6} \sum_{j=m}^{n-1} \frac{1}{2^j}.$$

Since $\sum_{j=0}^{\infty} 2^{-j}$ is a convergent geometric series, its sequence of partial sums is a Cauchy sequence. Combining this with the inequality above and Theorem 6.2.5, we see that (f_n) converges uniformly on $[0,1]$.

(c) It is straightforward to argue by induction that each f_n is a continuous increasing function satisfying $f(0) = 0$ and $f(1) = 1$. It follows from the Continuous Limit Theorem (Theorem 6.2.6), Lemma L.15, and the uniform convergence $f_n \to f$ that f is a continuous increasing function satisfying $f(0) = 0$ and $f(1) = 1$.

Let $x \in [0,1] \setminus C$ be given. By De Morgan's Laws we have

$$[0,1] \setminus C = [0,1] \setminus \left(\bigcap_{m=1}^{\infty} C_m \right) = \bigcup_{m=1}^{\infty} ([0,1] \setminus C_m).$$

Thus $x \in [0,1] \setminus C_m$ for some $m \in \mathbf{N}$. We constructed the sequence (f_n) in such a way that f_n is constant on the open set $[0,1] \setminus C_m$ for all $n \ge m$; the uniform convergence $f_n \to f$ then implies that f is constant on $[0,1] \setminus C_m$ for any $m \in \mathbf{N}$. The openness of $[0,1] \setminus C_m$ implies that there is some open interval I contained in $[0,1] \setminus C_m$ and containing x such that f is constant on I. It follows that that f is differentiable at x and moreover that $f'(x) = 0$.

Exercise 6.2.13. Recall that the Bolzano-Weierstrass Theorem (Theorem 2.5.5) states that every bounded sequence of real numbers has a convergent subsequence. An analogous statement for bounded sequences of functions is not true in general, but under stronger hypotheses several different conclusions are possible. One avenue is to assume the common domain for all of the functions in the sequence is countable. (Another is explored in the next two exercises.)

Let $A = \{x_1, x_2, x_3, ...\}$ be a countable set. For each $n \in \mathbf{N}$, let f_n be defined on A and assume there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in A$. Follow these steps to show that there exists a subsequence of (f_n) that converges pointwise on A.

(a) Why does the sequence of real numbers $f_n(x_1)$ necessarily contain a convergent subsequence (f_{n_k})? To indicate that the subsequence of functions (f_{n_k}) is generated by considering the values of the functions at x_1, we will use the notation $f_{n_k} = f_{1,k}$.

(b) Now, explain why the sequence $f_{1,k}(x_2)$ contains a convergent subsequence.

(c) Carefully construct a nested family of subsequences $(f_{m,k})$, and show how this can be used to produce a single subsequence of (f_n) that converges at every point of A.

Solution. For the purposes of this exercise, let us adopt some more precise, if cumbersome, notation for sequences. A sequence in a non-empty set E is a function $a : \mathbf{N} \to E$. A sequence $b : \mathbf{N} \to E$ is a subsequence of a if there exists a strictly increasing function $\theta : \mathbf{N} \to \mathbf{N}$ such that $b = a \circ \theta$, i.e. such that $b(n) = a(\theta(n))$ for all $n \in \mathbf{N}$. We shall write $b \triangleleft a$ to mean that b is a subsequence of a. Given this definition, it is clear that if c is a subsequence of b and if b is a subsequence of a, then c is a subsequence of a. In other words, $\triangleleft$ is transitive.

(a) Define $a_0 : \mathbf{N} \to \mathbf{R}^A$ (where $\mathbf{R}^A$ is the collection of all functions from A to $\mathbf{R}$) by $a_0(n) = f_n$. By assumption, the sequence of real numbers whose n^{th} term is $[a_0(n)](x_1) = f_n(x_1)$ is bounded. According to the Bolzano-Weierstrass Theorem there exists a strictly increasing function $\theta_1 : \mathbf{N} \to \mathbf{N}$ such that

$$\lim_{n\to\infty} [a_0(\theta_1(n))](x_1) = \lim_{n\to\infty} f_{\theta_1(n)}(x_1) = y_1$$

for some $y_1 \in \mathbf{R}$. Define $a_1 : \mathbf{N} \to \mathbf{R}^A$ by $a_1 = a_0 \circ \theta_1$. Note that $a_1 \triangleleft a_0$ and that

$$\lim_{n\to\infty} [a_1(n)](x_1) = \lim_{n\to\infty} f_{\theta_1(n)}(x_1) = y_1.$$

(b) The sequence of real numbers whose n^{th} term is $[a_1(n)](x_2) = f_{\theta_1(n)}(x_2)$ is bounded by assumption. The Bolzano-Weierstrass Theorem implies the existence of a strictly increasing function $\theta_2 : \mathbf{N} \to \mathbf{N}$ such that

$$\lim_{n\to\infty} [a_1(\theta_2(n))](x_2) = \lim_{n\to\infty} f_{\theta_1(\theta_2(n))}(x_2) = y_2$$

for some $y_2 \in \mathbf{R}$. Define $a_2 : \mathbf{N} \to \mathbf{R}^A$ by $a_2 = a_1 \circ \theta_2 = a_0 \circ \theta_1 \circ \theta_2$ and note that $a_2 \lhd a_1 \lhd a_0$. Note further that

$$\lim_{n\to\infty} [a_2(n)](x_2) = \lim_{n\to\infty} f_{\theta_1(\theta_2(n))}(x_2) = y_2$$

$$\text{and} \quad \lim_{n\to\infty} [a_2(n)](x_1) = \lim_{n\to\infty} f_{\theta_1(\theta_2(n))}(x_1) = y_1,$$

since subsequences of convergent sequences converge to the same limit as the parent sequence.

(c) We continue in this manner, obtaining for each $m \in \mathbf{N}$ a sequence $a_m : \mathbf{N} \to \mathbf{R}^A$, a strictly increasing function $\theta_m : \mathbf{N} \to \mathbf{N}$ such that $a_m = a_{m-1} \circ \theta_m = a_0 \circ \theta_1 \circ \cdots \circ \theta_m$, and a real number y_m such that

$$\lim_{n\to\infty} [a_m(n)](x_m) = \lim_{n\to\infty} f_{(\theta_1\circ\cdots\circ\theta_m)(n)}(x_m) = y_m.$$

It follows that $a_m \lhd a_{m-1} \lhd \cdots \lhd a_1 \lhd a_0$, which implies that

$$\lim_{n\to\infty} [a_m(n)](x_k) = \lim_{n\to\infty} f_{(\theta_1\circ\cdots\circ\theta_m)(n)}(x_k) = y_k$$

for each $k \in \{1, \ldots, m\}$, since subsequences of convergent sequences converge to the same limit as the parent sequence.

Define $\Theta : \mathbf{N} \to \mathbf{N}$ by $\Theta(n) = (\theta_1 \circ \cdots \circ \theta_n)(n)$; we claim that Θ is strictly increasing. Let $m < n$ be positive integers and observe that

$$m < n \leq \theta_n(n) \leq \cdots \leq (\theta_{m+1} \circ \cdots \circ \theta_n)(n),$$

where we have used that $n \leq \theta(n)$ for any strictly increasing function $\theta : \mathbf{N} \to \mathbf{N}$. Any composition of strictly increasing functions is a strictly increasing function; it follows that $\theta_1 \circ \cdots \circ \theta_m$ is a strictly increasing function and hence that

$$m < (\theta_{m+1} \circ \cdots \circ \theta_n)(n) \quad \Rightarrow \quad (\theta_1 \circ \cdots \circ \theta_m)(m) < (\theta_1 \circ \cdots \circ \theta_m \circ \cdots \circ \theta_n)(n),$$

i.e. $\Theta(m) < \Theta(n)$, as claimed.

Define $b : \mathbf{N} \to \mathbf{R}^A$ by $b = a_0 \circ \Theta$, so that

$$b(n) = (a_0 \circ \Theta)(n) = (a_0 \circ \theta_1 \circ \cdots \circ \theta_n)(n) = a_n(n).$$

This is a subsequence of a_0 since Θ is a strictly increasing function. This subsequence is sometimes known as the "diagonal subsequence"; the following visualization can explain why.

$$\begin{array}{c|cccc} a_1 & {\color{red}a_1(1)} & a_1(2) & a_1(3) & \cdots \\ a_2 & a_2(1) & {\color{red}a_2(2)} & a_2(3) & \cdots \\ a_3 & a_3(1) & a_3(2) & {\color{red}a_3(3)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The m^{th} row corresponds to the sequence a_m; note that each row is a subsequence of the row preceding it. The sequence b is obtained by taking the diagonal elements of this infinite array, highlighted in red.

Our goal now is to show that $b = (f_{\Theta(n)})_{n=1}^{\infty}$ converges pointwise on A to the function $f : A \to \mathbf{R}$ given by $f(x_m) = y_m$. Let $m \in \mathbf{N}$ be given and note that for $n \geq m+1$ we have

$$b(n) = (a_0 \circ \Theta)(n) = (a_0 \circ \theta_1 \circ \cdots \circ \theta_n)(n) = (a_m \circ \theta_{m+1} \circ \cdots \circ \theta_n)(n) = (a_m \circ \Theta_m)(n),$$

where $\Theta_m : \{m+1, m+2, ...\} \to \mathbf{N}$ is defined by

$$\Theta_m(n) = (\theta_{m+1} \circ \cdots \circ \theta_n)(n).$$

Similarly to how we showed that Θ is strictly increasing, we can show that Θ_m is strictly increasing. It follows that b is eventually a subsequence of a_m and hence that

$$\lim_{n\to\infty} [b(n)](x_m) = \lim_{n\to\infty} [a_m(n)](x_m) = y_m = f(x_m).$$

Exercise 6.2.14. A sequence of functions (f_n) defined on a set $E \subseteq \mathbf{R}$ is called *equicontinuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \mathbf{N}$ and $|x - y| < \delta$ in E.

(a) What is the difference between saying that a sequence of functions (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?

(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?

Solution.

(a) If (f_n) is equicontinuous then for a given $\varepsilon > 0$ the $\delta > 0$ that we obtain depends only on ε; if instead we only have that each f_n is individually uniformly continuous, then the δ may depend on n. In symbols, (f_n) is equicontinuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n \in \mathbf{N})((x, y \in E \text{ and } |x - y| < \delta) \quad \Rightarrow \quad |f_n(x) - f_n(y)| < \varepsilon),$$

whereas each f_n is individually uniformly continuous if

$$(\forall n \in \mathbf{N})(\forall \varepsilon > 0)(\exists \delta > 0)((x, y \in E \text{ and } |x - y| < \delta) \quad \Rightarrow \quad |f_n(x) - f_n(y)| < \varepsilon);$$

notice the order of the quantifiers.

(b) The issue occurs near 1; no matter how small δ is taken, it is possible to take n large enough and x within δ of 1 such that $f_n(x)$ and $f_n(1)$ are far apart. Geometrically, the slope of f_n gets very steep near 1 as we increase n. To be more precise, let $\delta > 0$ be given, take $n \in \mathbf{N}$ such that $\frac{1}{n} < \delta$, and let $x = 1 - \frac{1}{n}$. Observe that $1 - x < \delta$ and that

$$|f_n(1) - f_n(x)| = 1 - \left(1 - \tfrac{1}{n}\right)^n \geq 1 - e^{-1} > 0,$$

where we have used that $\left(1 - \frac{1}{n}\right)^n$ is an increasing sequence which converges to e^{-1}. Each g_n is uniformly continuous on $[0,1]$ by Theorem 4.4.7.

Exercise 6.2.15 (Arzela-Ascoli Theorem). For each $n \in \mathbf{N}$, let f_n be a function defined on $[0,1]$. If (f_n) is bounded on $[0,1]$—that is, there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in [0,1]$—and if the collection of functions (f_n) is equicontinuous (Exercise 6.2.14), follow these steps to show that (f_n) contains a uniformly convergent subsequence.

(a) Use Exercise 6.2.13 to produce a subsequence (f_{n_k}) that converges at every rational point in $[0,1]$. To simplify the notation, set $g_k = f_{n_k}$. It remains to show that (g_k) converges uniformly on all of $[0,1]$.

(b) Let $\varepsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that

$$|g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$$

for all $|x - y| < \delta$ and $k \in \mathbf{N}$. Using this δ, let $r_1, r_2, \ldots, r_m$ be a *finite* collection of rational points with the property that the union of the neighborhoods $V_\delta(r_i)$ contains $[0,1]$.

(c) Explain why there must exist an $N \in \mathbf{N}$ such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\varepsilon}{3}$$

for all $s, t \geq N$ and r_i in the finite subset of $[0,1]$ just described. Why does having the set $\{r_1, r_2, \ldots, r_m\}$ be finite matter?

(d) Finish the argument by showing that, for an arbitrary $x \in [0,1]$,

$$|g_s(x) - g_t(x)| < \varepsilon$$

for all $s, t \geq N$.

Solution.

(a) Since $\mathbf{Q} \cap [0,1]$ is countable, Exercise 6.2.13 implies the existence of the desired subsequence (g_k).

(b) Consider the open cover $[0,1] \subseteq \bigcup_{r \in \mathbf{Q} \cap [0,1]} V_\delta(r)$. Because $[0,1]$ is compact, there must exist a finite subcover, i.e. there must exist rationals $r_1, r_2, \ldots, r_m$ in $\mathbf{Q} \cap [0,1]$ such that $V_\delta(r_1) \cup \cdots \cup V_\delta(r_m)$ contains $[0,1]$.

(c) Let $i \in \{1, \ldots, m\}$ be given. Since (g_k) converges at every rational point in $[0,1]$, the sequence $(g_k(r_i))$ must be a Cauchy sequence. It follows that there exists an $N_i \in \mathbf{N}$ such that

$$s, t \geq N_i \quad \Rightarrow \quad |g_s(r_i) - g_t(r_i)| < \tfrac{\varepsilon}{3}.$$

Thus the desired $N \in \mathbf{N}$ is $N = \max\{N_1, ..., N_M\}$; the finiteness of $\{r_1, ..., r_m\}$ ensures this maximum exists.

(d) Let $x \in [0, 1]$ be given, so that $x \in V_\delta(r_i)$ for some $i \in \{1, ..., m\}$, and let $s, t \geq N$ be given. Observe that

$$|g_s(x) - g_t(x)| \leq |g_s(x) - g_s(r_i)| + |g_t(x) - g_t(r_i)| + |g_s(r_i) - g_t(r_i)| = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

where we have used that $|x - r_i| < \delta$. Theorem 6.2.5 allows us to conclude that (g_k) is a uniformly convergent subsequence of (f_n).

6.3. Uniform Convergence and Differentiation

Exercise 6.3.1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.

(b) Now, show that (g_n') converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g_n'$ and compare h and g'. Are they the same?

Solution.

(a) The limit function $g : [0,1] \to \mathbf{R}$ is given by $g(x) = 0$. For any $x \in [0,1]$ we have

$$|g_n(x) - g(x)| = \frac{x^n}{n} \leq \frac{1}{n};$$

it follows that the convergence $g_n \to g$ is uniform. Certainly g is differentiable on $[0,1]$ and satisfies $g'(x) = 0$ for each $x \in [0,1]$.

(b) The sequence (g_n') is given by $g_n'(x) = x^{n-1}$ for $x \in [0,1]$. This sequence converges pointwise to the function $h : [0,1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

The convergence cannot be uniform since each g_n' is continuous at 1 but h is not. Note that $h \neq g'$; this gives an alternative proof for showing that the convergence $g_n' \to h$ is not uniform, as uniform convergence $g_n' \to h$ would imply that $g' = h$ by Theorem 6.3.1/6.3.3.

Exercise 6.3.2. Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

(a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbf{R}$.

(b) Note that each h_n is differentiable. Show $g(x) = \lim h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is *not* uniform on any neighborhood of zero.

Solution.

(a) The pointwise limit is the function $h : \mathbf{R} \to \mathbf{R}$ given by $h(x) = \sqrt{x^2} = |x|$. For any $x \in \mathbf{R}$ we have

$$|h_n(x) - h(x)| = \sqrt{x^2 + n^{-1}} - \sqrt{x^2} = \frac{n^{-1}}{\sqrt{x^2 + n^{-1}} + \sqrt{x^2}} \le \frac{n^{-1}}{n^{-1/2}} = \frac{1}{\sqrt{n}}.$$

Thus the convergence $h_n \to h$ is uniform on $\mathbf{R}$.

(b) Observe that $h_n' : \mathbf{R} \to \mathbf{R}$ is given by

$$h_n'(x) = \frac{x}{\sqrt{x^2 + n^{-1}}}.$$

This sequence converges pointwise to the function $g : \mathbf{R} \to \mathbf{R}$ given by

$$g(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

The convergence $h_n' \to g$ cannot be uniform on any neighbourhood of zero each h_n' is continuous at zero but g is not. Alternatively, if the convergence $h_n' \to g$ were uniform then Theorem 6.3.1/6.3.3 would imply that h is differentiable at zero—but h fails to be differentiable precisely at zero.

Exercise 6.3.3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Find the points on $\mathbf{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbf{R}$. What is the limit function?

(b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Solution.

(a) From the observation

$$\frac{1}{2\sqrt{n}} - \frac{x}{1 + nx^2} = \frac{nx^2 - 2\sqrt{n}x + 1}{2\sqrt{n}(1 + nx^2)} = \frac{\left(\sqrt{n}x - 1\right)^2}{2\sqrt{n}(1 + nx^2)} \ge 0$$

we can see that $0 \le f_n(x) \le \frac{1}{2\sqrt{n}}$ for all $x \ge 0$ and also that $f_n(x) = \frac{1}{2\sqrt{n}}$ precisely when $x = \frac{1}{\sqrt{n}}$. Combining this with the fact that each f_n is an odd function, we see that

$$-\frac{1}{2\sqrt{n}} \le f_n(x) \le \frac{1}{2\sqrt{n}}$$

for every $x \in \mathbf{R}$ and furthermore that

$$f_n(x) = -\frac{1}{2\sqrt{n}} \quad \Leftrightarrow \quad x = -\frac{1}{\sqrt{n}} \quad \text{and} \quad f_n(x) = \frac{1}{2\sqrt{n}} \quad \Leftrightarrow \quad x = \frac{1}{\sqrt{n}}.$$

The bound $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$ for all $x \in \mathbf{R}$ shows that $f_n \to 0$ uniformly on $\mathbf{R}$.

(b) The quotient rule gives us

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

For $x \neq 0$ we have

$$f_n'(x) = \frac{\frac{1}{n^2x^4} - \frac{1}{nx^2}}{\left(\frac{1}{nx^2} + 1\right)^2} \to 0 \text{ as } n \to \infty,$$

and for $x = 0$ we have $f_n'(0) = 1$. In part (a) we showed that the limit function $f : \mathbf{R} \to \mathbf{R}$ was given by $f(x) = 0$. Thus $f'(x) = \lim f_n'(x) = 0$ for all $x \neq 0$ and $f'(0) = 0 \neq 1 = \lim f_n'(0)$.

Exercise 6.3.4. Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbf{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbf{R}$.

Solution. The bound

$$|h_n(x)| \leq \frac{1}{\sqrt{n}}$$

for each $x \in \mathbf{R}$ shows that $h_n \to 0$ uniformly on $\mathbf{R}$. The sequence of derivatives (h_n') is given by

$$a_n := h_n'(x) = \sqrt{n}\cos(nx).$$

We claim that (a_n) does not converge for any $x \in \mathbf{R}$; to see this, we will consider three cases.

Case 1. Suppose $x = k\pi$ for some even integer k. In this case we have $a_n = \sqrt{n}$, which diverges.

Case 2. Suppose $x = k\pi$ for some odd integer k. In this case we have $a_n = (-1)^n\sqrt{n}$, which diverges.

Case 3. Suppose x is not of the form $k\pi$ for any integer k and suppose by way of contradiction that $a_n \to L$ for some $L \in \mathbf{R}$. It follows that

$$\frac{a_n}{\sqrt{n}} = \cos(nx) \to 0,$$

which also implies that $\cos((n+1)x) \to 0$. Consider the trigonometric identity

$$\sin(nx) = \frac{\cos(nx)\cos(x) - \cos((n+1)x)}{\sin(x)};$$

since $x \neq k\pi$ for any integer k, we are not dividing by zero. Because both $\cos(nx) \to 0$ and $\cos((n+1)x) \to 0$, we see that $\sin(nx) \to 0$. It follows that

$$\sin^2(nx) + \cos^2(nx) \to 0,$$

which contradicts that $\sin^2(nx) + \cos^2(nx) = 1$ for each $n \in \mathbf{N}$.

Exercise 6.3.5. Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbf{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Solution.

(a) For a fixed $x \in \mathbf{R}$ we have

$$g_n(x) = \frac{x}{2} + \frac{x^2}{2n} \to \frac{x}{2} \text{ as } n \to \infty.$$

It follows that $g(x) = \frac{x}{2}$ and hence that $g'(x) = \frac{1}{2}$ for any $x \in \mathbf{R}$.

(b) The sequence of derivatives (g_n') is given by

$$g_n'(x) = \frac{1}{2} + \frac{x}{n}.$$

For $x \in [-M, M]$ we have

$$\left| g_n'(x) - \frac{1}{2} \right| = \frac{|x|}{n} \leq \frac{M}{n}.$$

Thus $g_n' \to \frac{1}{2}$ uniformly on any interval of the form $[-M, M]$. Observe that $0 \in [-M, M]$ and $g_n(0) = 0$ is convergent. We may now apply Theorem 6.3.3 to see that $g_n \to g$ uniformly on $[-M, M]$ and furthermore that $g'(x) = \lim g_n'(x) = \frac{1}{2}$ for any $x \in [-M, M]$. By taking M sufficiently large, this shows that $g'(x) = \frac{1}{2}$ for all $x \in \mathbf{R}$.

(c) The sequence (f_n) is given by

$$f_n(x) = \frac{nx^2 + 1}{2n + x}.$$

(Strictly speaking this is only defined on $\mathbf{R} \setminus \{-2n\}$, but since we are only interested in the limit as $n \to \infty$, this isn't a problem: eventually the sequence is defined on any interval of the form $[-M, M]$.)

Note that

$$f_n(x) = \frac{x^2 + \frac{1}{n}}{2 + \frac{x}{n}} \to \frac{x^2}{2} \text{ as } n \to \infty.$$

Thus the pointwise limit function is $f(x) = \frac{x^2}{2}$, which satisfies $f'(x) = x$.

The sequence of derivatives (f_n') is given by

$$f_n'(x) = \frac{nx^2 + 4n^2x - 1}{x^2 + 4nx + 4n^2} = \frac{\frac{x^2}{n} + 4x - \frac{1}{n^2}}{\frac{x^2}{n^2} + \frac{4x}{n} + 4} \to x \text{ as } n \to \infty.$$

For any $x \in [-M, M]$ observe that

$$|f_n'(x) - x| = \frac{\left|x^3 + 3nx^2 + 1\right|}{(2n + x)^2} \leq \frac{M^3 + 3M^2n + 1}{(2n - M)^2},$$

provided $n > \frac{M}{2}$. Note that the numerator of this bound is linear in n whereas the denominator is quadratic in n; it follows that this bound converges to zero and hence that $f_n' \to x$ uniformly on $[-M, M]$. Observe that $0 \in [-M, M]$ and $f_n(0) = \frac{1}{2n} \to 0$ as $n \to \infty$. Theorem 6.3.3 then implies that $f_n \to f$ uniformly on $[-M, M]$ and furthermore that $f'(x) = \lim f_n'(x) = x$ for any $x \in [-M, M]$. By taking M sufficiently large, this shows that $f'(x) = x$ for all $x \in \mathbf{R}$.

Exercise 6.3.6. Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbf{R}$.

(a) A sequence (f_n) of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

(b) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbf{R}$.

(c) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n') converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Solution.

(a) Define a sequence $(f_n : \mathbf{R} \to \mathbf{R})$ by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Observe that $f_n \to 0$ uniformly on $\mathbf{R}$ since $|f_n(x)| \leq \frac{1}{n}$ for any $x \in \mathbf{R}$. Certainly the zero function is differentiable everywhere, but each f_n is nowhere continuous and hence nowhere differentiable.

(b) Define a sequence $(f_n : \mathbf{R} \to \mathbf{R})$ by

$$f_n(x) = n.$$

Each f_n is differentiable and the sequence (f_n') is given by $f_n'(x) = 0$, which converges uniformly to the zero function. However, $(f_n(x))$ is divergent for every $x \in \mathbf{R}$.

(c) This is impossible. Any point $x \in \mathbf{R}$ is contained in some interval of the form $[-M, M]$; applying Theorem 6.3.3 to this interval shows that f is differentiable at x.

Exercise 6.3.7. Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that, for any $x \in [a, b]$ and $m, n \in \mathbf{N}$,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|.$$

Solution. Let $\varepsilon > 0$ be given. Since the sequence $(f_n(x_0))$ is convergent, there exists an $N_1 \in \mathbf{N}$ such that

$$n, m \geq N_1 \quad \Rightarrow \quad |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2},$$

and since the sequence (f_n') converges uniformly on $[a, b]$, there exists an $N_2 \in \mathbf{N}$ such that

$$x \in [a, b] \text{ and } n, m \geq N_2 \quad \Rightarrow \quad |f_n'(x) - f_m'(x)| < \frac{\varepsilon}{2(b-a)}.$$

Let $N = \max\{N_1, N_2\}$ and suppose that $n, m \geq N$ and $x \in (x_0, b]$ (the argument is easily modified if $x \in [a, x_0)$). Note that $f_n - f_m$ is differentiable on the interval $[x_0, x]$; the Mean Value Theorem then implies that there is some $c \in (x_0, x)$ such that

$$|x - x_0||f_n'(c) - f_m'(c)| = |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))|.$$

It follows that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\
&= |x - x_0||f_n'(c) - f_m'(c)| + |f_n(x_0) - f_m(x_0)| \\
&\leq (b-a)|f_n'(c) - f_m'(c)| + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

We have now shown that for any $n, m \geq N$ and $x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \varepsilon;$$

it follows from Theorem 6.2.5 that the sequence (f_n) is uniformly convergent on $[a, b]$.

6.4. Series of Functions

Exercise 6.4.1. Supply the details for the proof of the Weierstrass M-Test (Corollary 6.4.5).

Solution. Let $\varepsilon > 0$ be given. Since the series $\sum_{n=1}^{\infty} M_n$ is convergent, its sequence of partial sums is a Cauchy sequence. Consequently, there exists an $N \in \mathbf{N}$ such that

$$n > m \geq N \quad \Rightarrow \quad M_{m+1} + \cdots + M_n < \varepsilon.$$

Suppose $x \in A$ and $n > m \geq N$ and observe that

$$\left|f_{m+1}(x) + \cdots + f_n(x)\right| \leq \left|f_{m+1}(x)\right| + \cdots + \left|f_n(x)\right| \leq M_{m+1} + \cdots + M_n < \varepsilon.$$

It follows from Theorem 6.4.4 that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Exercise 6.4.2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution.

(a) This is true. Suppose that each g_n is defined on some domain $A \subseteq \mathbf{R}$. Note that Theorem 6.4.4 implies in particular that for any $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$x \in A \text{ and } n \geq N \quad \Rightarrow \quad |g_n(x)| \leq \varepsilon.$$

Thus g_n converges uniformly to the zero function.

(b) This is true. Suppose that each f_n and each g_n is defined on some domain $A \subseteq \mathbf{R}$. Theorem 6.4.4 implies that for any $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$x \in A \text{ and } n > m \geq N \quad \Rightarrow \quad g_{m+1}(x) + \cdots + g_n(x) < \varepsilon;$$

note we have used the non-negativity of each g_n. Suppose $x \in A$ and $n > m \geq N$. By assumption we have

$$f_{m+1}(x) + \cdots + f_n(x) \leq g_{m+1}(x) + \cdots + g_n(x) < \varepsilon.$$

By combining this inequality with the non-negativity of each f_n and Theorem 6.4.4, we see that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

(c) This is false. For each $n \in \mathbf{N}$ define a function $f_n : \mathbf{R} \to \mathbf{R}$ by $f_n(n) = \frac{1}{n}$ and $f_n(x) = 0$ for $x \neq n$. Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\sum_{n=1}^{\infty} f_n$ converges to f uniformly on $\mathbf{R}$. Observe that the partial sum function is

$$s_n(x) = f_1(x) + \cdots + f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \{1, \ldots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $|s_n(x) - f(x)| \leq \frac{1}{n+1}$ for any $x \in \mathbf{R}$. Because this bound converges to zero and does not depend on x, our claim follows.

Now observe that $\sup\{|f_n(x)| : x \in \mathbf{R}\} = \frac{1}{n}$ for any $n \in \mathbf{N}$; it follows that any bound M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ must satisfy $M_n \geq \frac{1}{n}$. Since the harmonic series diverges, it must be the case that $\sum_{n=1}^{\infty} M_n$ diverges. Thus the converse of the Weierstrass M-Test does not hold.

Exercise 6.4.3.

(a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbf{R}$.

(b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

Solution.

(a) Observe that

$$\left| \frac{\cos(2^n x)}{2^n} \right| \leq \frac{1}{2^n}$$

for every $x \in \mathbf{R}$. Since the series $\sum_{n=0}^{\infty} 2^{-n}$ is convergent, the Weierstrass M-Test implies that $g(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(2^n x)$ converges uniformly on $\mathbf{R}$. Each function $2^{-n} \cos(2^n x)$ is continuous on $\mathbf{R}$ and thus g is continuous on $\mathbf{R}$ by Theorem 6.4.2.

(b) To use Theorem 6.4.3, we would need to show that the series

$$\sum_{n=0}^{\infty} \left(\frac{\cos(2^n x)}{2^n} \right)' = -\sum_{n=0}^{\infty} \sin(2^n x)$$

converges uniformly on $\mathbf{R}$. However, this series does not even converge pointwise on $\mathbf{R}$. For example, consider the series of real numbers

$$\sum_{n=0}^{\infty} \sin(2^n).$$

To show that this series is divergent, we will show that the sequence $(\sin(2^n))$ does not converge to zero. To see this, consider the following two cases.

Case 1. If there exists an $N \in \mathbf{N}$ such that $\left|\sin(2^{n+1})\right| > |\sin(2^n)|$ for all $n \geq N$, then it must be the case that $\sin(2^n)$ does not converge to zero.

Case 2. If there does not exist such an N then there must be infinitely many $n \in \mathbf{N}$ such that $\left|\sin(2^{n+1})\right| \leq |\sin(2^n)|$. For such an n, the identity

$$\sin(2^{n+1}) = 2\sin(2^n)\cos(2^n)$$

and the fact that $\sin(2^n) \neq 0$ for any $n \in \mathbf{N}$ shows that $|\cos(2^n)| \leq \frac{1}{2}$. The Pythagorean identity then implies that $|\sin(2^n)| \geq \frac{\sqrt{3}}{2}$. So in this case the sequence $(\sin(2^n))$ satisfies $|\sin(2^n)| \geq \frac{\sqrt{3}}{2}$ infinitely often and hence does not converge to zero.

Thus Theorem 6.4.3 does not allow us to conclude anything about the differentiability of g.

Exercise 6.4.4. Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1 + x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Solution. For $|x| = 1$ we have $g(x) = \sum_{n=0}^{\infty} \frac{1}{2}$, which diverges. For $|x| > 1$ we have

$$\frac{x^{2n}}{1 + x^{2n}} = \frac{1}{x^{-2n} + 1} \to 1 \text{ as } n \to \infty$$

and thus $g(x)$ diverges.

Now suppose that $r > 0$ is such that $0 < r^2 < 1$ and observe that for all $x \in [-r, r]$ we have

$$0 \leq \frac{x^{2n}}{1 + x^{2n}} \leq x^{2n} \leq r^{2n}.$$

Since $\sum_{n=0}^{\infty} r^{2n}$ is a convergent geometric series, the Weierstrass M-Test implies that g converges uniformly on $[-r, r]$. Since any $x \in (-1, 1)$ is contained inside an interval of this form, we see that g converges and is continuous at each $x \in (-1, 1)$ by Theorem 6.4.2. Combining this with our previous discussion, we may conclude that g converges pointwise precisely on the open interval $(-1, 1)$.

Exercise 6.4.5.

(a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Solution.

(a) For any $x \in [-1, 1]$ we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the Weierstrass M-Test implies that h converges uniformly on $[-1, 1]$ and Theorem 6.4.2 then implies that h is continuous on $[-1, 1]$, since each function $\frac{x^n}{n^2}$ is continuous on $[-1, 1]$.

(b) Observe that

$$\left|\frac{x^n}{n}\right| \leq |x_0|^n$$

for every $x \in [-x_0, x_0]$. Since $\sum_{n=1}^{\infty} |x_0|^n$ is a convergent geometric series, the Weierstrass M-Test implies that f converges uniformly on $[-x_0, x_0]$. Theorem 6.4.2 then implies that f is continuous on $[-x_0, x_0]$ and in particular at x_0, since each function $\frac{x^n}{n}$ is continuous on $[-x_0, x_0]$.

Exercise 6.4.6. Let

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \frac{1}{x+4} - \cdots.$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Solution. Observe that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}.$$

The term-by-term differentiated series is

$$-\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(x+n)^2}.$$

Notice that

$$\left|\frac{(-1)^{n+1}}{(x+n)^2}\right|\leq\frac{1}{n^2}$$

for any $x\in(0,\infty)$ and any $n\in\mathbf{N}$. Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, the Weierstrass M-Test implies that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(x+n)^2}$ converges uniformly on $(0,\infty)$. It follows that the term-by-term differentiated series converges uniformly on $(0,\infty)$. Observe that

$$f(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{1+n}$$

converges by the Alternating Series Test (Theorem 2.7.7). Since any $x\in(0,\infty)$ is contained either inside an interval of the form $[a,1]$ or inside an interval of the form $[1,a]$, Theorem 6.4.3 allows us to conclude that f is defined and differentiable (hence continuous) at each $x\in(0,\infty)$.

Exercise 6.4.7. Let

$$f(x)=\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

Solution.

(a) Let $f_k:\mathbf{R}\to\mathbf{R}$ be given by $f_k(x)=\frac{\sin(kx)}{k^3}$ and observe that

$$|f_k'(x)|=\left|\frac{\cos(kx)}{k^2}\right|\leq\frac{1}{k^2}$$

for any $x\in\mathbf{R}$. The Weierstrass M-Test then implies that the series

$$\sum_{k=1}^{\infty}f_k'(x)=\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2}$$

converges uniformly on $\mathbf{R}$; since each f_k' is continuous on $\mathbf{R}$, Theorem 6.4.2 shows that $\sum_{k=1}^{\infty}f_k'(x)$ is also continuous on $\mathbf{R}$. Combining our previous discussion with Theorem 6.4.3 and the fact that $f(0)=0$, we see that $\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^3}$ converges uniformly on $\mathbf{R}$ to a differentiable function f, that

$$f'(x)=\sum_{k=1}^{\infty}f_k'(x)=\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2},$$

and that f' is continuous on $\mathbf{R}$.

(b) We will show that Theorem 6.4.3 cannot be used to determine if f is twice-differentiable on $\mathbf{R}$, by showing that the series of second derivatives

$$\sum_{k=1}^{\infty} f_k''(x) = -\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

does not converge uniformly on $\mathbf{R}$. To see this, we will use the negation of Theorem 6.4.4. Let $N \in \mathbf{N}$ be given and let $x = \frac{\pi}{4N}$. For any $N+1 \leq k \leq 2N$ we have

$$\frac{\pi}{4} \leq kx \leq \frac{\pi}{2} \quad \Rightarrow \quad \sin(kx) \geq \frac{1}{\sqrt{2}}.$$

Now observe that

$$\left| \sum_{k=N+1}^{2N} \frac{\sin(kx)}{k} \right| \geq \frac{1}{\sqrt{2}} \sum_{k=N+1}^{2N} \frac{1}{k} \geq \frac{1}{\sqrt{2}} \sum_{k=N+1}^{2N} \frac{1}{2N} = \frac{1}{2\sqrt{2}}.$$

It follows from Theorem 6.4.4 that the convergence of the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ is not uniform on $\mathbf{R}$. Consequently, we may not use Theorem 6.4.3 to conclude anything about the twice-differentiability of f on $\mathbf{R}$.

Exercise 6.4.8. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Solution. Let $f_k : \mathbf{R} \to \mathbf{R}$ be given by $f_k(x) = \frac{\sin(x/k)}{k}$, so that $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Observe that

$$f_k'(x) = \frac{\cos\left(\frac{x}{k}\right)}{k^2} \quad \text{and} \quad f_k''(x) = -\frac{\sin\left(\frac{x}{k}\right)}{k^3}.$$

The bound $|f_k''(x)| \leq \frac{1}{k^3}$ for all $x \in \mathbf{R}$ combined with the Weierstrass M-Test shows that the seires $\sum_{k=1}^{\infty} f_k''(x)$ converges uniformly on $\mathbf{R}$. Since

$$f(0) = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} f_k'(0) = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

are both convergent, Theorem 6.4.3 shows that $\sum_{k=1}^{\infty} f_k'(x)$ and $\sum_{k=1}^{\infty} f_k(x)$ both converge uniformly on $\mathbf{R}$. Furthermore,

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x) \quad \text{and} \quad f''(x) = \sum_{k=1}^{\infty} f_k''(x).$$

In particular, f is defined and continuous on $\mathbf{R}$.

Exercise 6.4.9. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function defined on all of $\mathbf{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

Solution.

(a) We have the bound

$$\frac{1}{x^2 + n^2} \leq \frac{1}{n^2}$$

for all $x \in \mathbf{R}$; the Weierstrass M-Test then implies that the series $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly on $\mathbf{R}$. Since each $\frac{1}{x^2+n^2}$ is continuous on $\mathbf{R}$, Theorem 6.4.2 allows us to conclude that h is also continuous on $\mathbf{R}$.

(b) The term-by-term differentiated series is

$$-\sum_{n=1}^{\infty} \frac{2x}{\left(x^2 + n^2\right)^2}.$$

Note that

$$|x| \leq 1 \text{ and } n \geq 2 \quad \Rightarrow \quad \left| \frac{2x}{\left(x^2 + n^2\right)^2} \right| \leq \frac{2}{n^4} \leq \frac{1}{n^2},$$

$$|x| > 1 \quad \Rightarrow \quad \left| \frac{2x}{\left(x^2 + n^2\right)^2} \right| = \frac{2|x|}{x^4 + 2x^2n^2 + n^4} \leq \frac{1}{|x|n^2} \leq \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and each summand $\frac{2x}{\left(x^2+n^2\right)^2}$ is continuous on $\mathbf{R}$, the Weierstrass M-Test and Theorem 6.4.2 imply that the series $\sum_{n=1}^{\infty} \frac{2x}{\left(x^2+n^2\right)^2}$ converges uniformly on $\mathbf{R}$ to a continuous function. We showed in part (a) that h converges uniformly on $\mathbf{R}$ and thus by Theorem 6.4.3 we have

$$h'(x) = -\sum_{n=1}^{\infty} \frac{2x}{\left(x^2 + n^2\right)^2}.$$

Exercise 6.4.10. Let $\{r_1, r_2, r_3, ...\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbf{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \leq r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbf{R}$ that is continuous at every irrational point.

Solution. Observe that $|u_n(x)| \leq 2^{-n}$ for all $x \in \mathbf{R}$. Since $\sum_{n=1}^{\infty} 2^{-n}$ is a convergent geometric series, the Weierstrass M-Test implies that h converges uniformly on $\mathbf{R}$. To see that h is strictly increasing, let $x < y$ be given real numbers. There is a countable infinity of rational numbers contained in $[x, y)$, which we can enumerate as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, ...\}$ of the sequence $\{r_1, r_2, r_3, ...\}$. Now,

$$h(y) - h(x) = \sum_{n=1}^{\infty} (u_n(y) - u_n(x)).$$

Let $n \in \mathbf{N}$ be given and consider the following three cases.

Case 1. If $r_n < x < y$ then $u_n(y) = u_n(x) = 2^{-n}$ and thus $u_n(y) - u_n(x) = 0$.

Case 2. If $x < y \leq r_n$ then $u_n(y) = u_n(x) = 0$ and thus $u_n(y) - u_n(x) = 0$.

Case 3. If $x \leq r_n < y$ then $r_n \in \{r_{n_1}, r_{n_2}, r_{n_3}, ...\}$, so that $n = n_k$ for some unique $k \in \mathbf{N}$. Thus $u_n(y) = 2^{-n_k}$ and $u_n(x) = 0$, which gives us $u_n(y) - u_n(x) = 2^{-n_k}$.

It follows that

$$h(y) - h(x) = \sum_{k=1}^{\infty} 2^{-n_k} > 0.$$

Thus $h(y) > h(x)$ whenever $y > x$, i.e. h is strictly increasing.

To see that h is continuous at every irrational point, let us first show that each u_n is continuous at every irrational point. Let $n \in \mathbf{N}$ and $y \in \mathbf{I}$ be given and let $\delta = |y - r_n|$; note that δ must be positive since y is not rational. There are two cases:

Case 1. If $y < r_n$ then $u_n(x) = 0$ for all $x \in (y - \delta, y + \delta)$ and hence u_n is continuous at y.

Case 2. If $y > r_n$ then $u_n(x) = 2^{-n}$ for all $x \in (y - \delta, y + \delta)$ and hence u_n is continuous at y.

Thus each summand u_n is continuous on $\mathbf{I}$, and we showed earlier that h converges uniformly on $\mathbf{R}$ and so in particular uniformly on $\mathbf{I}$; Theorem 6.4.2 allows us to conclude that h is also continuous on $\mathbf{I}$.

6.5. Power Series

Exercise 6.5.1. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

(a) Is g defined on $(-1,1)$? Is it continuous on this set? Is g defined on $(-1,1]$? Is it continuous on this set? What happens on $[-1,1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution.

(a) Observe that

$$g(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent by the Alternating Series Test (Theorem 2.7.7). It follows from Theorem 6.5.1 that g converges absolutely on $(-1,1)$ and hence g is defined on $(-1,1]$. Theorem 6.5.7 then implies that g is continuous on $(-1,1]$. Note that

$$g(-1) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$$

is the negated harmonic series, which diverges. Thus $g(-1)$ is not defined. We claim that $g(x)$ cannot possibly converge for any other points $|x| > 1$. To see this, suppose that $g(x)$ does converge for some $x \in \mathbf{R}$ such that $|x| > 1$ and let $r \in \mathbf{R}$ be such that $|x| > r > 1$. It follows from Theorem 6.5.1 that $g(r)$ converges absolutely—but

$$r + \frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4} + \cdots$$

diverges by comparison with the harmonic series.

(b) Theorem 6.5.7 guarantees that g is differentiable on $(-1,1)$ and the derivative is given by

$$g'(x) = 1 - x + x^2 - x^3 + x^4 - \cdots.$$

Note that this series does not converge at $x = 1$, despite $g(1)$ converging.

Exercise 6.5.2. Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbf{R}$.

(b) Diverges for every value of $x \in \mathbf{R}$.

(c) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.

(d) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.

(e) Converges conditionally at both $x = -1$ and $x = 1$.

Solution.

(a) Let $a_n = 0$ for every $n \geq 0$.

(b) This is impossible: any power series converges to zero at $x = 0$.

(c) Let $a_0 = 0$ and $a_n = \frac{1}{n^2}$ for each $n \in \mathbf{N}$. For $|x| \leq 1$ we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$$

and thus $\sum a_n x^n$ converges absolutely. If $|x| > 1$ then $n^{-2}x^n \to \infty$ and thus $\sum a_n x^n$ diverges.

(d) This is impossible. Note that

$$\sum_{n=0}^{\infty} |a_n(-1)^n| = \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} |a_n 1^n|.$$

Thus a power series converges absolutely at $x = 1$ if and only if it converges absolutely at $x = -1$.

(e) Let

$$a_n = \begin{cases} 0 & \text{if } n = 0 \text{ or } n \text{ is odd,} \\ \dfrac{2(-1)^{1+n/2}}{n} & \text{if } n \text{ is even,} \end{cases}$$

so that

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots.$$

Observe that

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n(-1)^n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

are both conditionally convergent series.

Exercise 6.5.3. Use the Weierstrass M-Test to prove Theorem 6.5.2.

Solution. Note that for any $x \in \mathbf{R}$ such that $|x| \leq |x_0|$ we have

$$|a_n x^n| = |a_n||x|^n \leq |a_n||x_0|^n.$$

The series $\sum_{n=0}^{\infty}|a_n||x_0|^n$ is convergent by assumption, so the Weierstrass M-Test implies that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$, where $c = |x_0|$.

Exercise 6.5.4 (Term-by-term Antidifferentiation). Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Solution.

(a) Let $x \in (-R, R)$ be given. Theorem 6.5.1 implies that the series $\sum_{n=0}^{\infty}|a_n||x|^n$ is convergent, which implies that the series $\sum_{n=0}^{\infty}|a_n||x|^{n+1}$ is convergent. Observe that

$$\left|\frac{a_n}{n+1} x^{n+1}\right| = \frac{|a_n|}{n+1}|x|^{n+1} \leq |a_n||x|^{n+1}$$

for each $n \geq 0$. Thus $F(x)$ is absolutely convergent by the Comparison Test. It follows that F is defined on the open interval $(-R, R)$ and it is then immediate from Theorem 6.5.7 that $F'(x) = f(x)$ on this interval.

(b) Corollary 5.3.4 implies that $g(x) = k + F(x)$ on $(-R, R)$ for some constant $k \in \mathbf{R}$. Thus

$$g(x) = k + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = k + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots.$$

Exercise 6.5.5.

(a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Solution.

(a) Certainly $0 < ns^{n-1}$ for each $n \geq 1$. Let $N \in \mathbf{N}$ be such that $s \leq 1 - \frac{1}{N+1}$. For $n \geq N$ it follows that

$$s \le 1 - \frac{1}{n+1} \quad \Leftrightarrow \quad (n+1)s \le n \quad \Leftrightarrow \quad (n+1)s^n \le ns^{n-1}.$$

Thus the sequence (ns^{n-1}) is bounded below and eventually decreasing. It follows from the Monotone Convergence Theorem that this sequence is convergent and hence bounded.

(b) From part (a), there is an $M > 0$ such that

$$n\left|\frac{x}{t}\right|^{n-1} \le M$$

for all $n \in \mathbf{N}$. Since $t \in (-R, R)$, Theorem 6.5.1 implies that the series $\sum_{n=0}^{\infty} a_n t^n$ is absolutely convergent. It follows that the series $\sum_{n=1}^{\infty} M|a_n|t^{n-1}$ is convergent. Now observe that

$$\left|na_n x^{n-1}\right| = n|a_n||x|^{n-1} = n\left|\frac{x}{t}\right|^{n-1} |a_n|t^{n-1} \le M|a_n|t^{n-1}$$

for each $n \in \mathbf{N}$. Thus by comparison with the convergent series $\sum_{n=1}^{\infty} M|a_n|t^{n-1}$ we see that the series $\sum_{n=1}^{\infty} na_n x^{n-1}$ is absolutely convergent. It follows that the power series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges on the open interval $(-R, R)$.

Exercise 6.5.6. Previous work on geometric series (Example 2.7.5) justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for all } |x| < 1.$$

Use the results about power series proved in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$. The discussion in Section 6.1 may be helpful.

Solution. The power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

has radius of convergence $R = 1$. Theorem 6.5.6 then implies that the formula

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

is valid on $(-1, 1)$, from which we obtain

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n \tag{1}$$

for all $x \in (-1, 1)$. Substituting $x = \frac{1}{2}$ gives us

$$2 = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Differentiating the power series (1) term-by-term gives us

$$\frac{1+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1},$$

valid on $(-1, 1)$, from which we obtain

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n,$$

valid on $(-1, 1)$. Substituting $x = \frac{1}{2}$ gives us

$$6 = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Exercise 6.5.7. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all $x \in (-1/L, 1/L)$. (The advice in Exercise 2.7.9 may be helpful.)

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbf{R}$.

(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n \quad \text{where } s_n = \sup\left\{ \left| \frac{a_{k+1}}{a_k} \right| : k \geq n \right\}.$$

(General properties of the *limit superior* are discussed in Exercise 2.4.7.)

Solution.

(a) Certainly the power series converges if $x = 0$, so suppose that $0 < |x| < \frac{1}{L}$. It follows that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = L|x| < 1$$

and hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent by the Ratio Test.

(b) Certainly the power series converges if $x = 0$, so suppose that $x \neq 0$. It follows that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = L|x| = 0$$

and hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent by the Ratio Test.

(c) Let us refine the Ratio Test (Exercise 2.7.9) as follows. Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, if

$$\lim_{n\to\infty} s_n = r < 1 \quad \text{where} \quad s_n = \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\},$$

then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. To see this, let r' be such that $r < r' < 1$. Since $\lim_{n\to\infty} s_n = r$, there is an $N \in \mathbf{N}$ such that

$$|s_N - r| = s_N - r < r' - r \quad \Rightarrow \quad s_N < r';$$

for the first equality we have used that the sequence (s_n) decreases to r (see Exercise 2.4.7). It follows from this inequality that

$$n \geq N \quad \Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| \leq s_N < r' \quad \Rightarrow \quad |a_{n+1}| < |a_n| r'.$$

We may now argue as in Exercise 2.7.9 to conclude the proof of this refined ratio test. Using this refined test, the desired results about power series follow as in parts (a) and (b).

Exercise 6.5.8.

(a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Solution.

(a) Let us show that if a power series $h(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $h(x) = 0$ for all $x \in (-R, R)$, then $a_n = 0$ for all $n \geq 0$. Theorem 6.5.7 implies that

$$h^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}$$

for all $x \in (-R, R)$ and all $k \geq 0$, where $h^{(k)}$ is the k^{th} derivative of h. Since h is identically zero on $(-R, R)$, it must be that $h^{(k)}$ is identically zero on $(-R, R)$ and thus

$$0 = h^{(k)}(0) = k! a_k \quad \Leftrightarrow \quad a_k = 0$$

for each $k \geq 0$.

Now suppose that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$. It follows that

$$\sum_{n=0}^{\infty}(a_n - b_n)x^n = 0$$

for all $x \in (-R, R)$ and our previous discussion then shows that $a_n - b_n = 0$ for all $n \geq 0$.

(b) Theorem 6.5.7 gives us

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = f'(x)$$

for all $x \in (-R, R)$. It follows from part (a) that

$$a_{n+1} = \frac{a_n}{n+1}$$

for all $n \geq 0$. From $f(0) = 1$ we obtain $a_0 = 1$ and hence $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}$, and in general

$$a_n = \frac{1}{n!}.$$

Exercise 6.5.9. Review the definitions and results from Section 2.8 concerning products of series and Cauchy products in particular. At the end of Section 2.9, we mentioned the following result: If both $\sum a_n$ and $\sum b_n$ converge conditionally to A and B respectively, then it is possible for the Cauchy product,

$$\sum d_n \quad \text{where} \quad d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

to diverge. However, if $\sum d_n$ does converge, then it must converge to AB. To prove this, set

$$f(x) = \sum a_n x^n, \quad g(x) = \sum b_n x^n, \quad \text{and} \quad h(x) = \sum d_n x^n.$$

Use Abel's Theorem and the result in Exercise 2.8.7 to establish this result.

Solution. Our hypothesis is that f, g, and h all converge at $x = 1$. It follows from Theorem 6.5.1 that f and g converge absolutely for any $x \in (-1, 1)$ and hence by Exercise 2.8.7 we have

$$h(x) = \sum_{n=0}^{\infty} d_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = f(x)g(x) \quad \text{for all } x \in (-1, 1). \tag{1}$$

Abel's Theorem (Theorem 6.5.4) implies that f, g, and h converge uniformly on $[0, 1]$ and hence are continuous on $[0, 1]$. The continuity at $x = 1$ allows us to extend the equality in (1) to all $x \in (-1, 1]$, which gives us

$$h(1) = \sum_{n=0}^{\infty} d_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = AB.$$

Exercise 6.5.10. Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-R, R)$, and assume $(x_n) \to 0$ with $x_n \neq 0$. If $g(x_n) = 0$ for all $n \in \mathbf{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

Solution. Theorem 6.5.7 implies that g is continuous at zero. It follows that

$$b_0 = g(0) = g\Big(\lim_{k\to\infty} x_k\Big) = \lim_{k\to\infty} g(x_k) = 0.$$

Theorem 6.5.7 also allows us to differentiate g term-by-term, obtaining the power series

$$g'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1},$$

valid on $(-R, R)$. It follows that

$$b_1 = g'(0) = \lim_{k\to\infty} \frac{g(x_k) - g(0)}{x_k} = 0.$$

We can continue in this manner to see that $b_n = 0$ for each $n \geq 0$, which by Exercise 6.5.8 implies that g is identically zero on $(-R, R)$.

Exercise 6.5.11. A series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel-summable to L* if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0, 1)$ and $L = \lim_{x\to 1^-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty} (-1)^n$ is Abel-summable and find the sum.

Solution.

(a) Suppose $\sum_{n=0}^{\infty} a_n$ converges to L. In other words, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges to L at $x = 1$; Abel's Theorem then implies that the power series is uniformly convergent on $[0, 1]$ and hence continuous on $[0, 1]$. It follows that

$$\lim_{x\to 1^-} f(x) = f\Big(\lim_{x\to 1^-} x\Big) = f(1) = \sum_{n=0}^{\infty} a_n = L.$$

(b) Let

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x};$$

this is valid for $|x| < 1$. It follows that $\sum_{n=0}^{\infty} (-1)^n$ is Abel-summable to $\frac{1}{2}$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{1+x} = \frac{1}{2}.$$

6.6. Taylor Series

Exercise 6.6.1. The derivation in Example 6.6.1 shows the Taylor series for $\arctan(x)$ is valid for all $x \in (-1, 1)$. Notice, however, that the series also converges when $x = 1$. Assuming that $\arctan(x)$ is continuous, explain why the value of the series at $x = 1$ must necessarily be $\arctan(1)$. What interesting identity do we get in this case?

Solution. The power series

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

converges on $(-1, 1]$; it follows from Theorem 6.5.7 that the power series is continuous on this interval. Given that $\arctan$ is also continuous at $x = 1$, it follows that the function $f : (-1, 1] \to \mathbf{R}$ given by

$$f(x) = \arctan(x) - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)$$

is continuous at $x = 1$ and satisfies $f(x) = 0$ for all $x \in (-1, 1)$. The continuity at $x = 1$ then implies that $f(1) = 0$ also, which gives us the identity

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Exercise 6.6.2. Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

(a) $x\cos(x^2)$

(b) $x/(1 + 4x^2)^2$

(c) $\log(1 + x^2)$

Solution.

(a) Starting from the power series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

which converges for any $x \in \mathbf{R}$, Theorem 6.5.6 implies that the differentiated series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

also converges for any $x \in \mathbf{R}$. From this we obtain

$$x\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \cdots,$$

valid for all $x \in \mathbf{R}$.

(b) Starting from the power series

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots,$$

derived in Example 6.6.1 and valid for all $|x| < 1$, we obtain

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots,$$

valid for all $|x| < 1$. Substituting $4x^2$ for x gives us

$$\frac{1}{(1+4x^2)^2} = 1 - 2 \cdot 4x^2 + 3 \cdot 4^2 x^4 - 4 \cdot 4^3 x^6 + 5 \cdot 4^4 x^8 - \cdots,$$

valid for all $\left|4x^2\right| < 1$, i.e. all $|x| < \frac{1}{2}$. From this we obtain

$$\begin{aligned}\frac{x}{(1+4x^2)^2} &= \sum_{n=0}^{\infty} (-1)^n (n+1) 4^n x^{2n+1} \\ &= x - 2 \cdot 4x^3 + 3 \cdot 4^2 x^5 - 4 \cdot 4^3 x^7 + 5 \cdot 4^4 x^9 - \cdots,\end{aligned}$$

valid for all $|x| < \frac{1}{2}$. Note that for $x = \frac{1}{2}$ the power series becomes

$$\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+1),$$

which is divergent. Similarly, $x = -\frac{1}{2}$ gives us the divergent series

$$-\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+1).$$

Thus the power series representation

$$\begin{aligned}\frac{x}{(1+4x^2)^2} &= \sum_{n=0}^{\infty} (-1)^n (n+1) 4^n x^{2n+1} \\ &= x - 2 \cdot 4x^3 + 3 \cdot 4^2 x^5 - 4 \cdot 4^3 x^7 + 5 \cdot 4^4 x^9 - \cdots\end{aligned}$$

is valid precisely on the open interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(c) Starting from the power series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots,$$

valid on $(-1, 1)$, we may use Exercise 6.5.4 to take term-by-term antiderivatives:

$$\log(1+x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n};$$

this is valid for $x \in (-1, 1)$. Taking $x = 0$ shows that $C = 0$. Note that the power series $\sum_{n=1}^{\infty} n^{-1}(-1)^{n+1}x^n$ is convergent at $x = 1$ and divergent at $x = -1$. Thus the power series converges precisely on $(-1, 1]$. It follows from Abel's Theorem (Theorem 6.5.4) and the continuity of $\log(1 + x)$ at $x = 1$ that the power series representation

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

is valid on the half-open interval $(-1, 1]$. Since $x^2 \in [0, 1] \subseteq (-1, 1]$ whenever $x \in [-1, 1]$, we see that the representation

$$\log(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \frac{x^{12}}{6} + \cdots$$

is valid on $[-1, 1]$.

Exercise 6.6.3. Derive the formula for the Taylor coefficients given in Theorem 6.6.2.

Solution. Suppose $f : (-R, R) \to \mathbf{R}$, for some $R > 0$, is infinitely differentiable and has a power series representation

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Theorem 6.5.7 implies that on the interval $(-R, R)$ we have

$$f^{(n)}(x) = \sum_{k=0}^{\infty}(k + 1) \cdots (k + n)a_{k+n}x^k,$$

from which it is immediate that $f^{(n)}(0) = n!a_n$.

Exercise 6.6.4. Explain how Lagrange's Remainder Theorem can be modified to prove

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log(2).$$

Solution. Let $f : (0, \infty) \to \mathbf{R}$ be given by $f(x) = \log(x)$. Note that f is infinitely differentiable and satisfies

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n - 1)!}{x^n}$$

for each $n \geq 1$. Consider the Taylor series of f centred at $a = 1$:

$$f(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x - 1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x - 1)^n.$$

As noted in the textbook (p. 202), Lagrange's Remainder Theorem implies that for each $N \in \mathbf{N}$ there exists some $c_N \in (1, 2)$ such that

$$E_N(2) = \log(2) - \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} = \frac{f^{N+1}(c_N)}{(N+1)!} = \frac{(-1)^N}{(N+1)c_N^{N+1}}.$$

This implies that $|E_N(2)| \le (N+1)^{-1}$, from which it follows that $\lim_{N\to\infty} E_N(2) = 0$, i.e.

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Exercise 6.6.5.

(a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.

(b) Verify the formula $f'(x) = e^x$.

(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

Solution.

(a) Since $f^{(n)}(x) = e^x$ for any $n \ge 0$, the Taylor coefficients are

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{n!}.$$

Let $R > 0$ be given and suppose $x \in [-R, R]$. For $N \in \mathbf{N}$, Lagrange's Remainder Theorem implies that there is some c_N satisfying $|c_N| < |x| \le R$ and

$$|E_N(x)| = \left| \frac{f^{N+1}(c_N)}{(N+1)!} x^{N+1} \right| = \frac{e^{c_N}}{(N+1)!} |x|^{N+1} \le \frac{e^R R^{N+1}}{(N+1)!}.$$

Since $\lim_{N\to\infty} e^R R^{N+1} [(N+1)!]^{-1} = 0$ (this can be seen using, for example, Stirling's approximation), we see that the Taylor series converges uniformly to e^x on $[-R, R]$.

(b) Differentiating the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

term-by-term gives us

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x) = e^x.$$

(c) Informally,

$$e^x \cdot e^{-x} = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 - x + \frac{x^2}{2!} - \cdots\right)$$

$$= 1 + (1-1)x + \left(\frac{1}{2!} + \frac{1}{2!} - 1\right)x^2 + \left(\frac{1}{3!} - \frac{1}{3!} + \frac{1}{2!} - \frac{1}{2!}\right)x^3 + \cdots = 1.$$

Exercise 6.6.6. Review the proof that $g'(0) = 0$ for the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

introduced at the end of this section.

(a) Compute $g'(x)$ for $x \neq 0$. Then use the definition of the derivative to find $g''(0)$.

(b) Compute $g''(x)$ and $g'''(x)$ for $x \neq 0$. Use these observations and invent whatever notation is needed to give a general description for the nth derivative $g^{(n)}(x)$ at points different from zero.

(c) Construct a general argument for why $g^{(n)}(0) = 0$ for all $n \in \mathbf{N}$.

Solution.

(a) For $x \neq 0$ we have

$$g'(x) = \left(e^{-x^{-2}}\right)' = 2x^{-3}e^{-x^{-2}}.$$

This gives us

$$g''(0) = \lim_{x\to 0} \frac{g'(x) - g'(0)}{x} = 2\lim_{x\to 0} \frac{x^{-4}}{e^{x^{-2}}}.$$

Note that

$$\lim_{x\to 0} \frac{\left(x^{-4}\right)'}{\left(e^{x^{-2}}\right)'} = \lim_{x\to 0} \frac{-4x^{-5}}{-2x^{-3}e^{x^{-2}}} = 2\lim_{x\to 0} \frac{x^{-2}}{e^{x^{-2}}}.$$

Note further that

$$\lim_{x\to 0} \frac{\left(x^{-2}\right)'}{\left(e^{x^{-2}}\right)'} = \lim_{x\to 0} \frac{-2x^{-3}}{-2x^{-3}e^{x^{-2}}} = \lim_{x\to 0} e^{-x^{-2}} = 0.$$

It follows from two applications of the ∞/∞ case of L'Hospital's Rule (Theorem 5.3.8) that $g''(0) = 0$.

(b) For $x \neq 0$ we have

$$g''(x) = 4x^{-6}e^{-x^{-2}} - 6x^{-4}e^{-x^{-2}} \text{ and } g''(x) = 8x^{-9}e^{-x^{-2}} - 36x^{-7}e^{-x^{-2}} + 24x^{-5}e^{-x^{-2}}.$$

We conjecture that for $x \neq 0$ the n^{th} derivative of g is given by the formula

$$g^{(n)}(x) = e^{-x^{-2}} \sum_{j=0}^{n-1} c_{n,j} x^{-3n+2j},$$

where $c_{n,0}, \ldots, c_{n,n-1}$ are real numbers depending only on n. We will prove this by induction. The base case $n = 1$ was handled in part (a). Suppose the result is true for some $n \in \mathbf{N}$. For $x \neq 0$, the usual rules of differentiation give us

$$\begin{aligned} g^{(n+1)}(x) &= -2x^{-3}e^{-x^{-2}} \sum_{j=0}^{n-1} c_{n,j} x^{-3n+2j} + e^{-x^{-2}} \sum_{j=0}^{n-1} (-3n+2j) c_{n,j} x^{-3n+2j-1} \\ &= e^{-x^{-2}} \left[\sum_{j=0}^{n-1} -2c_{n,j} x^{-3(n+1)+2j} + \sum_{j=0}^{n-1} (-3n+2j) c_{n,j} x^{-3n+2j-1} \right] \\ &= e^{-x^{-2}} \left[\sum_{j=0}^{n-1} -2c_{n,j} x^{-3(n+1)+2j} + \sum_{j=1}^{n} (-3n+2j-2) c_{n,j-1} x^{-3(n+1)+2j} \right] \\ &= e^{-x^{-2}} \sum_{j=0}^{n} c_{n+1,j} x^{-3(n+1)+2j}, \end{aligned}$$

where

$$c_{n+1,j} = \begin{cases} -2c_{n,0} & \text{if } j = 0, \\ -2c_{n,j} + (-3n+2j-2)c_{n,j-1} & \text{if } 1 \leq j \leq n-1, \\ (-n-2)c_{n,n-1} & \text{if } j = n. \end{cases}$$

This completes the induction step and the proof.

(c) Using L'Hospital's Rule, it is straightforward to argue that

$$\lim_{x \to 0} x^{-j} e^{-x^{-2}} = 0$$

for any positive integer j. Using this result and part (b), for any $n \in \mathbf{N}$ we have

$$g^{(n+1)}(0) = \lim_{x \to 0} \frac{g^{(n)}(x)}{x} = \lim_{x \to 0} \left(\sum_{j=0}^{n-1} c_{n,j} x^{-3n+2j-1} e^{-x^{-2}} \right) = 0.$$

Exercise 6.6.7. Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.

(c) An infinitely differentiable function $f(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Solution.

(a) Consider $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = (1 + x^2)^{-1}$, which satisfies

$$g^{(n)}(x) = \frac{p_n(x)}{(1+x^2)^{2n}}$$

for $n \in \mathbf{N}$ and some polynomial p_n. As shown in Example 6.6.1, the Taylor series of g is

$$1 - x^2 + x^4 - x^6 + \cdots,$$

which converges if and only if $|x| < 1$.

(b) As shown in the textbook and Exercise 6.6.6, the function $g : \mathbf{R} \to \mathbf{R}$ given by

$$g(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has the same Taylor series as the zero function and yet satisfies $g(x) \neq 0$ for all $x \neq 0$. It follows that the function $h : \mathbf{R} \to \mathbf{R}$ given by

$$h(x) = \begin{cases} e^{-x^{-2}} + \sin(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has the same Taylor series as sin and yet satisfies $h(x) \neq \sin(x)$ for all $x \neq 0$.

(c) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Slight modifications to the arguments given in the textbook and Exercise 6.6.6 show that f is infinitely differentiable and satisfies $f^{(n)}(0) = 0$ for all $n \in \mathbf{N}$, so that each Taylor coefficient of f is zero, i.e. the Taylor series of f is zero. Since $f(x) = 0$ if and only if $x \leq 0$, we see that the Taylor series of f converges to f if and only if $x \leq 0$.

Exercise 6.6.8. Here is a weaker form of Lagrange's Remainder Theorem whose proof is arguably more illuminating than the one for the stronger result.

(a) First establish a lemma: If g and h are differentiable on $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ for all $t \in [0, x]$, then $g(t) \leq h(t)$ for all $t \in [0, x]$.

(b) Let f, S_N, and E_N be as Theorem 6.6.3, and take $0 < x < R$. If $\left|f^{N+1}(t)\right| \leq M$ for all $t \in [0, x]$, show

$$|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}.$$

Solution.

(a) It will suffice to show that if f is differentiable on $[0, x]$ with $f(0) = 0$ and $f'(t) \geq 0$ for all $t \in [0, x]$, then $f(t) \geq 0$ for all $t \in [0, x]$; the general case then follows by considering

$f = h - g$. Suppose therefore that $t \in [0, x]$ is given. Applying the Mean Value Theorem to f on the interval $[0, t]$ yields some $c \in (0, t)$ such that

$$f(t) - f(0) = f'(c)(t - 0) \quad \Leftrightarrow \quad f(t) = f'(c)t \geq 0.$$

(b) Take $g(t) = f^{(N)}(t) - f^{(N)}(0)$ and $h(t) = Mt$ in the lemma of part (a) to see that

$$f^{(N)}(t) - f^{(N)}(0) \leq Mt$$

for all $t \in [0, x]$. Using this result, take $g(t) = f^{(N-1)}(t) - \big(f^{(N-1)}(0) + f^{(N)}(0)t\big)$ and $h(t) = \frac{Mt^2}{2}$ in the lemma of part (a) to see that

$$f^{(N-1)}(t) - \big(f^{(N-1)}(0) + f^{(N)}(0)t\big) \leq \frac{Mt^2}{2}$$

for all $t \in [0, x]$. If we continue in this manner we obtain the inequality

$$f(t) - \left(f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \cdots + \frac{f^{(N)}(0)}{N!}t^N\right) = E_N(t) \leq \frac{Mt^{N+1}}{(N+1)!} \tag{1}$$

for all $t \in [0, x]$. Repeating this process, starting with $g(t) = -Mt$ and $h(t) = f^N(t) - f^N(0)$, we obtain

$$-\frac{Mt^{N+1}}{(N+1)!} \leq E_N(t) \tag{2}$$

for all $t \in [0, x]$. Taking $t = x$ in (1) and (2) gives us

$$|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}.$$

Exercise 6.6.9 (Cauchy's Remainder Theorem). Let f be differentiable $N+1$ times on $(-R, R)$. For each $a \in (-R, R)$, let $S_N(x, a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x, a) = \sum_{n=0}^{N} c_n (x-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x, a) = f(x) - S_N(x, a)$. Now fix $x \neq 0$ in $(-R, R)$ and consider $E_N(x, a)$ as a function of a.

(a) Find $E_N(x, x)$.

(b) Explain why $E_N(x, a)$ is differentiable with respect to a, and show

$$E_N'(x, a) = \frac{-f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Show

$$E_N(x) = E_N(x, 0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

Solution.

(a) We have

$$E_N(x, x) = f(x) - S_N(x, x) = f(x) - c_0 = f(x) - f(x) = 0.$$

(b) We are given that f is $N+1$ times differentiable on $(-R, R)$ and hence, by the usual rules of differentiation,

$$\begin{aligned}
E_N'(x, a) &= \frac{\mathrm{d}}{\mathrm{d}a}\left[f(x) - f(a) - \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right] \\
&= -f'(a) - \sum_{n=1}^{N} \frac{1}{n!} \frac{\mathrm{d}}{\mathrm{d}a}\left[f^{(n)}(a)(x-a)^n\right] \\
&= -f'(a) - \sum_{n=1}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} \\
&= \left[\frac{-f^{(N+1)}(a)}{N!}(x-a)^N - \sum_{n=1}^{N-1} \frac{f^{(n+1)}(a)}{n!}(x-a)^n\right] \\
&\qquad + \left[-f'(a) + \sum_{n=0}^{N-1} \frac{f^{(n+1)}(a)}{n!}(x-a)^n\right]
\end{aligned}$$

$$= \frac{-f^{(N+1)}(a)}{N!}(x-a)^N - \sum_{n=1}^{N-1} \frac{f^{(n+1)}(a)}{n!}(x-a)^n + \sum_{n=1}^{N-1} \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

$$= \frac{-f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Using the Mean Value Theorem on $E_N(x,a)$, as a function of a, on the interval $[0,x]$ yields some $c \in (0,x)$ such that

$$E_N(x,x) - E_N(x,0) = E'_N(x,c)x.$$

By parts (a) and (b) this expression becomes

$$E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x.$$

Exercise 6.6.10. Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0,1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem proved in Exercise 6.6.9 to show the series representation for f holds on $[0,1)$.

Solution.

(a) It is a straightforward calculation to see that

$$f^{(n)}(x) = \frac{(2n-1)(2n-3)\cdots(3)(1)}{2^n}(1-x)^{-n-1/2},$$

which gives us

$$f^{(n)}(0) = \frac{(2n-1)(2n-3)\cdots(3)(1)}{2^n}.$$

Thus the Taylor series for f centered at zero is

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)(2n-3)\cdots(3)(1)}{(2^n)(n!)}x^n = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)(2n-3)\cdots(3)(1)}{(2n)(2n-2)\cdots(2)(1)}x^n$$

$$= 1 + \sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} \frac{2j-1}{2j}\right)x^n.$$

For $0 < x < \frac{1}{2}$ and $n \geq 2$, Lagrange's Remainder Theorem states that there is some c_n such that $0 < c_n < x$ and

$$E_{n-1}(x) = \frac{f^{(n)}(c_n)}{n!}x^n = \left(\prod_{j=1}^{n} \frac{2j-1}{2j}\right)\left(\frac{x}{1-c_n}\right)^n \left(\frac{1}{\sqrt{1-c_n}}\right).$$

Note that $0 < c_n < x \le \frac{1}{2}$ implies that

$$\frac{1}{2} < 1 - c_n < 1 \quad \Rightarrow \quad 1 < \frac{1}{1-c_n} < 2 \quad \Rightarrow \quad 0 < \frac{x}{1-c_n} < 1$$

and thus $0 < \left(\frac{x}{1-c_n}\right)^n \left(\frac{1}{\sqrt{1-c_n}}\right) < \sqrt{2}$, which gives us

$$0 < E_{n-1}(x) < \sqrt{2}\left(\prod_{j=1}^{n} \frac{2j-1}{2j}\right).$$

We showed in Exercise 2.7.10 (b) that $\lim_{n\to\infty} \prod_{j=1}^{n} \frac{2j-1}{2j} = 0$. It now follows from the Squeeze Theorem that $\lim_{n\to\infty} E_{n-1}(x) = 0$. This argument breaks down when $x > \frac{1}{2}$ since in that case we can no longer conclude that the sequence $\left(\frac{x}{1-c_n}\right)^n \left(\frac{1}{\sqrt{1-c_n}}\right)$ is bounded.

(b) For $x \in (0, 1)$ and $n \in \mathbf{N}$, Cauchy's Remainder Theorem (see Exercise 6.6.9) states that there exists some c_n such that $0 < c_n < x$ and

$$\begin{aligned} E_n(x) &= \frac{f^{(n+1)}(c_n)}{n!}(x - c_n)^n \\ &= \frac{(2n+1)(2n-1)\cdots(3)(1)}{2^{n+1}n!}(1-c_n)^{-n-3/2}(x-c_n)^n x \\ &= \frac{x}{(1-c_n)^{3/2}}\left(\prod_{j=1}^{n} \frac{2j-1}{2j}\right)(n+1)\left(\frac{x-c_n}{1-c_n}\right)^n. \end{aligned}$$

Note that the inequalities $0 < c_n < x < 1$ imply that

$$\frac{x}{(1-c_n)^{3/2}} < \frac{x}{(1-x)^{3/2}} \quad \text{and} \quad \frac{x-c_n}{1-c_n} < x.$$

Note further that since $0 < \frac{2j-1}{2j} = 1 - \frac{1}{2j} < 1$ for each $1 \le j \le n$, the product $\prod_{j=1}^{n} \frac{2j-1}{2j}$ is strictly less than 1. It follows that

$$0 \le E_n(x) < \frac{x}{(1-x)^{3/2}}(n+1)x^n.$$

Since $\lim_{n\to\infty}(n+1)x^n = 0$ (which can be seen using, for example, L'Hospital's Rule), the Squeeze Theorem implies that $\lim_{n\to\infty} E_n(x) = 0$.

6.7. The Weierstrass Approximation Theorem

Exercise 6.7.1. Assuming WAT, show that if f is continuous on $[a, b]$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a, b]$.

Solution. The Weierstrass Approximation Theorem implies that for each $n \in \mathbf{N}$ there exists a polynomial p_n such that

$$|f(x) - p_n(x)| < \frac{1}{n}$$

for all $x \in [a, b]$. It follows that $p_n \to f$ uniformly on $[a, b]$.

Exercise 6.7.2. Prove Theorem 6.7.3.

Solution. Since f is a continuous function defined on the compact set $[a, b]$, Theorem 4.4.7 implies that f is uniformly continuous on $[a, b]$. Thus there exists a $\delta > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \delta$ and for each $0 \leq k \leq n$ let $x_k = a + k\frac{b-a}{n}$. Let $\phi : [a, b] \to \mathbf{R}$ be the polygonal function which is linear on each subinterval $[x_k, x_{k+1}]$ and passes through the points $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$. For $x \in [a, b]$ we have $x \in [x_k, x_{k+1}]$ for some $0 \leq k \leq n-1$. It follows that

$$|f(x) - \phi(x)| \leq |f(x) - \phi(x_k)| + |\phi(x_k) - \phi(x)| \leq |f(x) - \phi(x_k)| + \left|\phi(x_k) - \phi(x_{k+1})\right|;$$

for the last inequality we are using that ϕ is a line segment on the interval $[x_k, x_{k+1}]$ and thus $|\phi(x) - \phi(y)| \leq \left|\phi(x_k) - \phi(x_{k+1})\right|$ for any $x \in [x_k, x_{k+1}]$. By definition we have $\phi(x_k) = f(x_k)$ for any k and so

$$|f(x) - \phi(x)| \leq |f(x) - f(x_k)| + \left|f(x_k) - f(x_{k+1})\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Exercise 6.7.3.

(a) Find the second degree polynomial $p(x) = q_0 + q_1 x + q_2 x^2$ that interpolates the three points $(-1, 1), (0, 0)$, and $(1, 1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1, 1]$ on the same set of axes.

(b) Find the fourth degree polynomial that interpolates $g(x) = |x|$ at the points $x = -1, -1/2, 0, 1/2$, and 1. Add a sketch of this polynomial to the graph from (a).

Solution.

(a) It is clear that the desired second degree polynomial is $p(x) = x^2$.

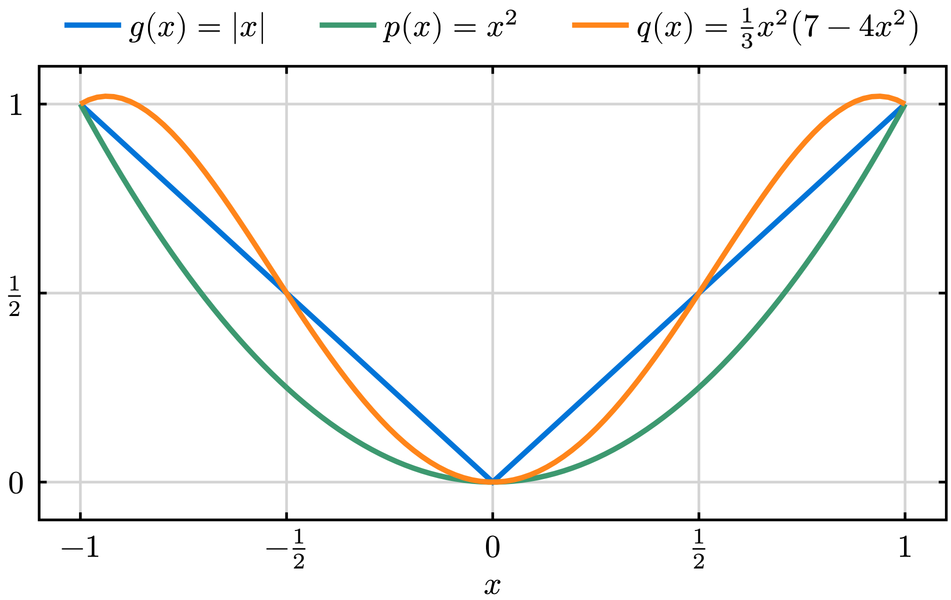
(b) We are looking for a polynomial $q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ such that

$$q(-1) = 1, \quad q\left(-\tfrac{1}{2}\right) = \tfrac{1}{2}, \quad q(0) = 0, \quad q\left(\tfrac{1}{2}\right) = \tfrac{1}{2}, \quad \text{and} \quad q(1) = 1.$$

The condition $q(0) = 0$ immediately gives us $a_0 = 0$ and the remaining four conditions give us the linear system

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

Solving this system, we obtain $a_1 = 0, a_2 = \frac{7}{3}, a_3 = 0$, and $a_4 = -\frac{4}{3}$. Thus the desired fourth degree polynomial is $q(x) = \frac{1}{3}x^2(7 - 4x^2)$.



Exercise 6.7.4. Show that $f(x) = \sqrt{1 - x}$ has Taylor series coefficients a_n where $a_0 = 1$ and

$$a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Solution. We have $f(0) = a_0 = 1$ and a straightforward calculation shows that

$$f^{(n)}(x) = \frac{-1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n}(1 - x)^{-n-1/2}$$

for $n \geq 1$. It follows from this that

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{-1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n n!} = \frac{-1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for any $n \geq 1$.

Exercise 6.7.5.

(a) Follow the advice in Exercise 6.6.9 to prove the Cauchy form of the remainder:

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x.

(b) Use this result to prove equation (1) is valid for all $x \in (-1, 1)$.

Solution.

(a) See Exercise 6.6.9.

(b) Suppose $0 < |x| < 1$. For $n \in \mathbf{N}$, the Cauchy Remainder Theorem implies that there exists some c_n such that $0 < |c_n| < |x|$ and

$$\begin{aligned} E_n(x) &= \frac{f^{(n+1)}(c_n)}{n!}(x-c_n)^n x \\ &= \frac{-1\cdot 3\cdots(2n-3)(2n-1)}{2^{n+1}n!}(1-c_n)^{-n-3/2}(x-c_n)^n x \\ &= -\frac{1}{2}\cdot\frac{1\cdot 3\cdots(2n-3)(2n-1)}{2\cdot 4\cdots(2n-2)(2n)}\left(\frac{x-c_n}{1-c_n}\right)^n \frac{x}{(1-c_n)^{3/2}} \\ &= -\frac{1}{2}\left(\prod_{j=1}^{n}\frac{2j-1}{2j}\right)\left(\frac{x-c_n}{1-c_n}\right)^n \frac{x}{(1-c_n)^{3/2}}. \end{aligned}$$

Since $\frac{2j-1}{2j} < 1$ for each $1 \le j \le n$, we have $\prod_{j=1}^{n}\frac{2j-1}{2j} < 1$ and thus

$$|E_n(x)| < \left|\frac{x-c_n}{1-c_n}\right|^n \frac{|x|}{(1-c_n)^{3/2}};$$

we have used that $|c_n| < 1$ implies $0 < 1-c_n$ to obtain $|1-c_n| = 1-c_n$. Note that

$$c_n \le |c_n| < |x| \quad\Rightarrow\quad -|x| < -c_n \quad\Rightarrow\quad \frac{1}{(1-c_n)^{3/2}} < \frac{1}{(1-|x|)^{3/2}}.$$

Note further that if $0 < c_n < x < 1$ then

$$xc_n < c_n \quad\Rightarrow\quad x - c_n < x - xc_n \quad\Rightarrow\quad \frac{x-c_n}{1-c_n} < x \quad\Rightarrow\quad \left|\frac{x-c_n}{1-c_n}\right| < |x|,$$

and if $-1 < x < c_n < 0$ then

$$c_n < xc_n \quad\Rightarrow\quad c_n - x < xc_n - x \quad\Rightarrow\quad \frac{c_n-x}{1-c_n} < -x \quad\Rightarrow\quad \left|\frac{x-c_n}{1-c_n}\right| < |x|.$$

Combining these inequalities, we see that

$$|E_n(x)| < \frac{|x|^{n+1}}{(1-|x|)^{3/2}}$$

and it follows that $\lim_{n\to\infty} E_n(x) = 0$ since $|x| < 1$.

Exercise 6.7.6.

(a) Let

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$. Show $c_n < \frac{2}{\sqrt{2n+1}}$.

(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients generated in Exercise 6.7.4.

(c) Carefully explain how this verifies that equation (1) holds for all $x \in [-1, 1]$.

Solution.

(a) We will prove this by induction. For the base case $n = 1$, we have

$$c_1 = \frac{1}{2} < \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{2(1)+1}}.$$

Now suppose that the inequality holds for some $n \in \mathbf{N}$, so that

$$c_{n+1} = c_n \cdot \frac{2n+1}{2n+2} < \frac{2}{\sqrt{2n+1}} \cdot \frac{2n+1}{2n+2} = \frac{2\sqrt{2n+1}}{2n+2}.$$

Observe that

$$\begin{aligned}
\frac{2\sqrt{2n+1}}{2n+2} < \frac{2}{\sqrt{2n+3}} \quad &\Leftrightarrow \quad \frac{\sqrt{2n+1}}{2n+2} < \frac{1}{\sqrt{2n+3}} \\
&\Leftrightarrow \quad \frac{2n+1}{4n^2+8n+4} < \frac{1}{2n+3} \\
&\Leftrightarrow \quad 4n^2+8n+3 < 4n^2+8n+4 \\
&\Leftrightarrow \quad 0 < 1.
\end{aligned}$$

Thus $c_{n+1} < \frac{2}{\sqrt{2n+3}}$. This completes the induction step and the proof.

(b) Since $\sum_{n=0}^{\infty} |a_n| = 1 + \sum_{n=1}^{\infty} |a_n|$, it will suffice to show that $\sum_{n=1}^{\infty} |a_n|$ is convergent. Note that for $n \geq 1$ we have, by part (a),

$$|a_n| = \frac{c_n}{2n-1} < \frac{2}{(2n-1)\sqrt{2n+1}} < \frac{2}{(2n-1)^{3/2}} \leq \frac{2}{n^{3/2}}.$$

Since the series $\sum_{n=1}^{\infty} 2n^{-3/2}$ is convergent by Corollary 2.4.7, we see by comparison that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(c) Part (b) shows that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at the points $x = -1$ and $x = 1$. It follows from Abel's Theorem (Theorem 6.5.4) that the power series converges uniformly and hence is continuous on $[-1, 1]$. Thus the function $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \sqrt{1-x} - \sum_{n=0}^{\infty} a_n x^n$$

is continuous on its domain and, by Exercise 6.7.5, satisfies $h(x) = 0$ for all $x \in (-1, 1)$. It must then be the case that $h(-1) = h(1) = 0$ also.

Exercise 6.7.7.

(a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\varepsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$||x| - q(x)| < \varepsilon$$

for all $x \in [-1, 1]$.

(b) Generalize this conclusion to an arbitrary interval $[a, b]$.

Solution.

(a) Note that $x \in [-1, 1]$ implies that $1 - x^2 \in [0, 1]$ and thus by Exercise 6.7.6 we have

$$\sqrt{1 - (1-x)^2} = \sum_{n=0}^{\infty} a_n (1-x^2)^n.$$

As we showed in Exercise 6.7.6 this convergence is uniform, so there exists an $N \in \mathbf{N}$ such that

$$\left| \sqrt{1 - (1-x)^2} - \sum_{n=0}^{N} a_n (1-x^2)^n \right| = \left| |x| - \sum_{n=0}^{N} a_n (1-x^2)^n \right| < \varepsilon$$

for all $x \in [-1, 1]$. Thus the desired polynomial is $q(x) = \sum_{n=0}^{N} a_n (1-x^2)^n$.

(b) For $a < b$ and $\varepsilon > 0$, we would like to find a polynomial p such that $||x| - p(x)| < \varepsilon$ for all $x \in [a, b]$. Let $c = \max\{|a|, |b|\}$ and note that $c > 0$. Note further that $x \in [a, b]$ implies that $\frac{x}{c} \in [-1, 1]$ and thus by part (a) there exists a polynomial q such that

$$\left| \left| \frac{x}{c} \right| - q\left(\frac{x}{c} \right) \right| < \frac{\varepsilon}{c} \tag{1}$$

for all $\frac{x}{c} \in [-1, 1]$, i.e. for all $x \in [-c, c]$. Let p be the polynomial given by $p(x) = cq\left(\frac{x}{c}\right)$. It follows from (1) that

$$||x| - p(x)| < \varepsilon$$

for all $x \in [-c, c]$ and hence in particular for all $x \in [a, b]$.

Exercise 6.7.8.

(a) Fix $a \in [-1, 1]$ and sketch

$$h_a(x) = \frac{1}{2}(|x - a| + (x - a))$$

over $[-1, 1]$. Note that h_a is polygonal and satisfies $h_a(x) = 0$ for all $x \in [-1, a]$.

(b) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1, 1]$.

(c) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1.$$

Show there exist constants $b_0, b_1, \ldots, b_{n-1}$ so that

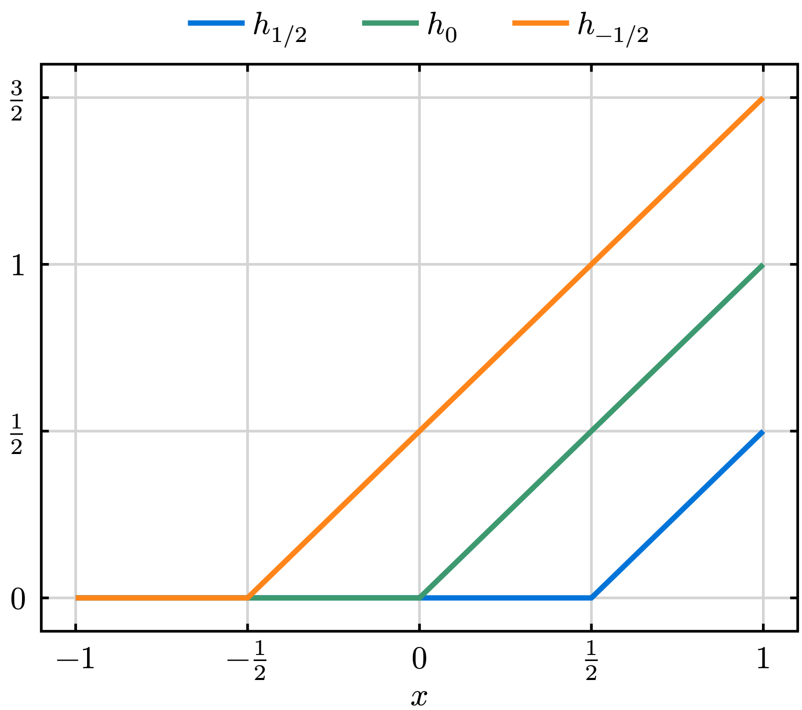
$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1, 1]$.

(d) Complete the proof of WAT for the interval $[-1, 1]$, and then generalize to an arbitrary interval $[a, b]$.

Solution.

(a) Below is a sketch of $h_{1/2}, h_0$, and $h_{-1/2}$ on $[-1, 1]$.



(b) From Exercise 6.7.7 (b), for a given $\varepsilon > 0$ we know that there exists a polynomial q such that

$$||x - a| - q(x - a)| < 2\varepsilon$$

for all $x \in [-1,1]$. Let $p(x) = \frac{1}{2}q(x-a) + \frac{1}{2}(x-a)$ and observe that

$$|h_a(x) - p(x)| = \frac{1}{2}||x-a| - q(x-a)| < \varepsilon$$

for all $x \in [-1,1]$.

(c) For $0 \leq j \leq n-1$, the polygonal function ϕ is given by a line segment on the subinterval $[a_j, a_{j+1}]$; let m_j be the slope of this line segment, i.e.

$$m_j = \frac{\phi(a_{j+1}) - \phi(a_j)}{a_{j+1} - a_j}.$$

Now set $b_0 = m_0$ and $b_j = m_j - m_{j-1}$ for $j \in \{1, \ldots, n-1\}$ and let $\psi : [-1,1] \to \mathbf{R}$ be given by

$$\psi(x) = \phi(a_0) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x).$$

Our aim is to show that $\phi(x) = \psi(x)$ for all $x \in [-1,1]$. For such an x, we have $x \in [a_j, a_{j+1}]$ for some $j \in \{1, \ldots, n-1\}$. Note that, since ϕ is linear on the subinterval $[a_j, a_{j+1}]$ with slope m_j,

$$\phi(x) = \phi(a_j) + m_j(x - a_j).$$

Note further that

$$h_{a_0}(x) = x - a_0, \quad \ldots, \quad h_{a_j}(x) = x - a_j, \quad \text{and} \quad h_{a_{j+1}} = \cdots = h_{a_{n-1}}(x) = 0.$$

Thus

$$\begin{aligned}
\psi(x) &= \phi(a_0) + b_0 h_{a_0}(x) + b_1 h_{a_1} + \cdots + b_j h_{a_j}(x) \\
&= \phi(a_0) + m_0(x - a_0) + (m_1 - m_0)(x - a_1) + \cdots + (m_j - m_{j-1})(x - a_j) \\
&= \underbrace{\phi(a_0) + m_0(a_1 - a_0)}_{\phi(a_1)} + m_1(a_2 - a_1) + \cdots + m_{j-1}(a_j - a_{j-1}) + m_j(x - a_j) \\
&= \underbrace{\phi(a_1) + m_1(a_2 - a_1)}_{\phi(a_2)} + \cdots + m_{j-1}(a_j - a_{j-1}) + m_j(x - a_j) \\
&= \cdots \\
&= \phi(a_j) + m_j(x - a_j) \\
&= \phi(x).
\end{aligned}$$

(d) Let $f : [-1,1] \to \mathbf{R}$ be continuous and let $\varepsilon > 0$ be given. By Theorem 6.7.3 (see Exercise 6.7.2), there exists a polygonal function $\phi : [-1,1] \to \mathbf{R}$ which is linear on each subinterval of some partition

$$-1 = a_0 < a_1 < \cdots < a_n = 1$$

and which satisfies $|f(x) - \phi(x)| < \frac{\varepsilon}{2}$ for all $x \in [-1,1]$. By part (c), there exist constants $b_0, \ldots, b_{n-1}$ such that

$$\phi(x) = \phi(a_0) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1, 1]$. Furthermore, by part (b), for each $j \in \{0, ..., n-1\}$ there exists a polynomial p_j such that

$$\left|h_{a_j}(x) - p_j(x)\right| < \frac{\varepsilon}{2n\left(1 + \left|b_j\right|\right)}.$$

Let p be the polynomial given by

$$p(x) = \phi(a_0) + b_0 p_0(x) + \cdots + b_{n-1} p_{n-1}(x)$$

and observe that for any $x \in [-1, 1]$ we have

$$\begin{aligned}
|\phi(x) - p(x)| &= \left|b_0 h_{a_0}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x) - b_0 p_0(x) - \cdots - b_{n-1} p_{n-1}(x)\right| \\
&\leq |b_0||h_{a_0}(x) - p_0(x)| + \cdots + |b_{n-1}||h_{a_{n-1}}(x) - p_{n-1}(x)| \\
&< \frac{\varepsilon|b_0|}{2n(1 + |b_0|)} + \cdots + \frac{\varepsilon|b_{n-1}|}{2n(1 + |b_{n-1}|)} \\
&< \frac{\varepsilon}{2n} + \cdots + \frac{\varepsilon}{2n} \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

It now follows that for any $x \in [-1, 1]$ we have

$$|f(x) - p(x)| \leq |f(x) - \phi(x)| + |\phi(x) - p(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We can now prove the general case. For $a < b$, let $f : [a, b] \to \mathbf{R}$ be continuous and let $\varepsilon > 0$ be given. We would like to find a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$. Note that the function

$$\begin{aligned}
[-1, 1] &\to [a, b] \\
x &\mapsto \frac{b - a}{2}(x + 1) + a
\end{aligned}$$

is a continuous bijection with continuous inverse

$$\begin{aligned}
[a, b] &\to [-1, 1] \\
x &\mapsto \frac{2(x - a)}{b - a} - 1.
\end{aligned}$$

Thus $g : [-1, 1] \to \mathbf{R}$ given by

$$g(x) = f\left(\frac{b - a}{2}(x + 1) + a\right)$$

is well-defined and, as a composition of continuous functions, is continuous on $[-1, 1]$. It follows from our previous discussion that there exists a polynomial q such that $|g(x) - q(x)| < \varepsilon$ for all $x \in [-1, 1]$. Let p be the polynomial defined by

$$p(x) = q\left(\frac{2(x-a)}{b-a} - 1\right).$$

Since $x \in [a,b]$ implies that $\frac{2(x-a)}{b-a} - 1 \in [-1,1]$, we have

$$\left| g\left(\frac{2(x-a)}{b-a} - 1\right) - q\left(\frac{2(x-a)}{b-a} - 1\right) \right| = |f(x) - p(x)| < \varepsilon$$

for all $x \in [a,b]$.

Exercise 6.7.9.

(a) Find a counterexample which shows that WAT is not true if we replace the closed interval $[a,b]$ with the open interval (a,b).

(b) What happens if we replace $[a,b]$ with the closed set $[a,\infty)$. Does the theorem still hold?

Solution.

(a) Consider $f : (0,1) \to \mathbf{R}$ given by $f(x) = x^{-1}$. Since any polynomial is bounded on $(0,1)$, if we could uniformly approximate f with a polynomial on $(0,1)$ then we would have that f is bounded on $(0,1)$, which is not true.

(b) The theorem does not hold. Consider $g : [0,\infty) \to \mathbf{R}$ given by $g(x) = \sin(x)$. Evidently g cannot be uniformly approximated by a constant polynomial on $[0,\infty)$, and for a non-constant polynomial p we have $\lim_{x\to\infty}|p(x)| = +\infty$, whereas $|g(x)| \leq 1$ for all $x \in [0,\infty)$ it follows that we cannot uniformly approximate g with a non-constant polynomial on $[0,\infty)$ either.

Exercise 6.7.10. Is there a countable subset of polynomials $\mathcal{C}$ with the property that every continuous function on $[a,b]$ can be uniformly approximated by polynomials from $\mathcal{C}$?

Solution. There is such a countable subset. Let $\mathcal{P}(\mathbf{R})$ be the collection of polynomials with real coefficients, let $\mathcal{P}(\mathbf{Q}) \subseteq \mathcal{P}(\mathbf{R})$ be the collection of polynomials with rational coefficients, and for each $n \geq 0$ let $\mathcal{P}_n(\mathbf{Q}) \subseteq \mathcal{P}(\mathbf{Q})$ be the collection of polynomials of degree n with rational coefficients. Observe that $\mathcal{P}_0(\mathbf{Q})$ is in bijection with $\mathbf{Q} \setminus \{0\}$ and $\mathcal{P}_n(\mathbf{Q})$ is in bijection with $(\mathbf{Q} \setminus \{0\}) \times \mathbf{Q}^{n-1}$ for each $n \geq 1$. Thus each $\mathcal{P}_n(\mathbf{Q})$ is countable and it follows from the expression

$$\mathcal{P}(\mathbf{Q}) = \{0\} \cup \bigcup_{n=0}^{\infty} \mathcal{P}_n(\mathbf{Q})$$

(by 0 we mean the zero polynomial) and Theorem 1.5.8 (ii) that $\mathcal{P}(\mathbf{Q})$ is countable.

Now let $a < b$ be given and let $M = \max\{|a|, |b|, 1\}$. Suppose $f : [a,b] \to \mathbf{R}$ is continuous and let $\varepsilon > 0$ be given. By the Weierstrass Approximation Theorem, there exists a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}(\mathbf{R})$$

such that $|f(x) - p(x)| < \frac{\varepsilon}{2}$ for all $x \in [a, b]$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, we can choose rational numbers $b_n, b_{n-1}, \ldots, b_1, b_0$ such that $\left|a_j - b_j\right| < \varepsilon(2M^n(n+1))^{-1}$ for each $j \in \{0, \ldots, n\}$. Let

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \in \mathcal{P}(\mathbf{Q})$$

and observe that for any $x \in [a, b]$ we have

$$\begin{aligned}
|p(x) - q(x)| &= \left|(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_1 - b_1)x + (a_0 - b_0)\right| \\
&\leq |a_n - b_n||x|^n + |a_{n-1} - b_{n-1}||x|^{n-1} + \cdots + |a_1 - b_1||x| + |a_0 - b_0| \\
&\leq |a_n - b_n|M^n + |a_{n-1} - b_{n-1}|M^{n-1} + \cdots + |a_1 - b_1|M + |a_0 - b_0| \\
&\leq |a_n - b_n|M^n + |a_{n-1} - b_{n-1}|M^n + \cdots + |a_1 - b_1|M^n + |a_0 - b_0|M^n \\
&\leq \frac{\varepsilon}{2(n+1)} + \frac{\varepsilon}{2(n+1)} + \cdots + \frac{\varepsilon}{2(n+1)} + \frac{\varepsilon}{2(n+1)} \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

It follows that

$$|f(x) - q(x)| \leq |f(x) - p(x)| + |p(x) - q(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for any $x \in [a, b]$. Thus the desired countable subset $\mathcal{C}$ is $\mathcal{P}(\mathbf{Q})$.

Exercise 6.7.11. Assume that f has a continuous derivative on $[a, b]$. Show that there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon \quad \text{and} \quad |f'(x) - p'(x)| < \varepsilon$$

for all $x \in [a, b]$.

Solution. By assumption f' is continuous on $[a, b]$, so the Weierstrass Approximation Theorem yields a polynomial q such that

$$|f'(x) - q(x)| < \min\left\{\varepsilon, \frac{\varepsilon}{b - a}\right\}$$

for all $x \in [a, b]$. Let p be the polynomial which satisfies $p' = q$ and $p(a) = f(a)$ and observe that $|f'(x) - p'(x)| = |f'(x) - q(x)| < \varepsilon$ for each $x \in [a, b]$. Now let $g : [a, b] \to \mathbf{R}$ be given by $g(x) = f(x) - p(x)$. Observe that $g(a) = 0$ and $g'(x) = f'(x) - q(x)$, so that $|g'(x)| < \varepsilon(b - a)^{-1}$ for all $x \in [a, b]$. Let $x \in (a, b]$ be given. By the Mean Value Theorem (Theorem 5.3.2), there exists some $c \in (a, x)$ such that

$$|f(x) - p(x)| = |g(x) - g(a)| = |g'(c)(x - a)| \leq \frac{\varepsilon}{b - a}(b - a) = \varepsilon.$$

Thus p is the desired polynomial.

Chapter 7. The Riemann Integral

7.2. The Definition of the Riemann Integral

Exercise 7.2.1. Let f be a bounded function on $[a, b]$ and let P be an arbitrary partition of $[a, b]$. First, explain why $U(f) \geq L(f, P)$. Now, prove Lemma 7.2.6.

Solution. Lemma 7.2.4 implies that $L(f, P)$ is a lower bound of the set $\{U(f, Q) : Q \in \mathcal{P}\}$ and thus $U(f) \geq L(f, P)$. Since P was an arbitrary partition of $[a, b]$, we have now shown that $U(f)$ is an upper bound of the set $\{L(f, P) : P \in \mathcal{P}\}$ and thus $U(f) \geq L(f)$.

Exercise 7.2.2. Consider $f(x) = 1/x$ over the interval $[1, 4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(a) Compute $L(f, P), U(f, P)$, and $U(f, P) - L(f, P)$.

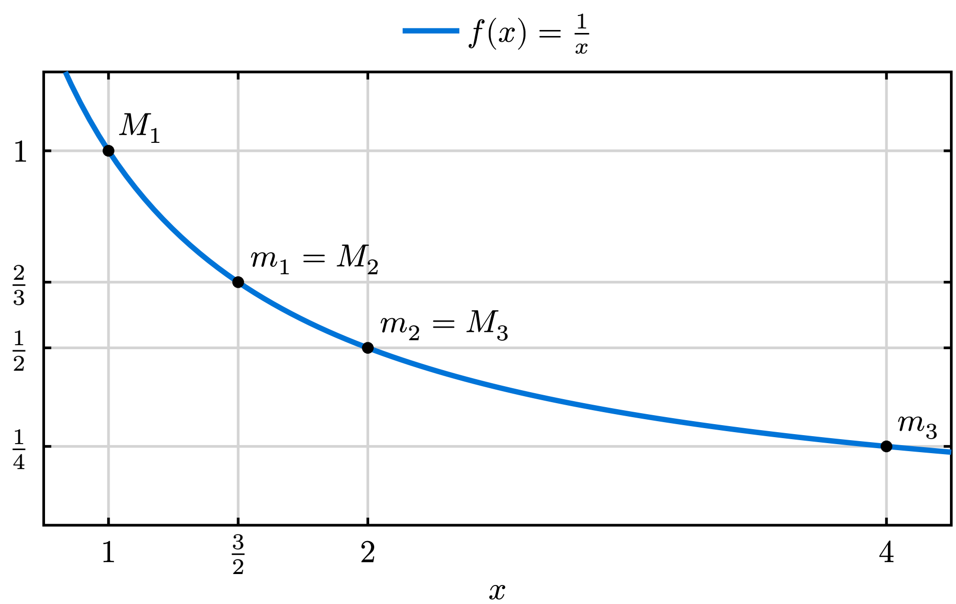
(b) What happens to the value of $U(f, P) - L(f, P)$ when we add the point 3 to the partition?

(c) Find a partition P' of $[1, 4]$ for which $U(f, P') - L(f, P') < 2/5$.

Solution.

(a) Since f is strictly decreasing over $[1, 4]$ we have

$$m_1 = f\left(\frac{3}{2}\right) = \frac{2}{3}, \quad m_2 = f(2) = \frac{1}{2}, \quad m_3 = f(4) = \frac{1}{4},$$
$$M_1 = f(1) = 1, \quad M_2 = f\left(\frac{3}{2}\right) = \frac{2}{3}, \quad M_3 = f(2) = \frac{1}{2}.$$



Thus $L(f,P)=\frac{13}{12}, U(f,P)=\frac{11}{6}$, and $U(f,P)-L(f,P)=\frac{3}{4}$.

(b) Letting $P=\left\{1,\frac{3}{2},2,3,4\right\}$, a similar calculation to part (a) shows that

$$U(f,P)-L(f,P)=\frac{1}{2}.$$

(c) Letting $P'=\left\{1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2,3,4\right\}$, a straightforward calculation shows that

$$U(f,P')-L(f,P')=\frac{3}{8}<\frac{2}{5}.$$

Exercise 7.2.3 (Sequential Criterion for Integrability).

(a) Prove that a bounded function f is integrable on $[a,b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0,$$

and in this case $\int_a^b f=\lim_{n\to\infty}U(f,P_n)=\lim_{n\to\infty}L(f,P_n)$.

(b) For each n, let P_n be the partition of $[0,1]$ into n equal subintervals. Find formulas for $U(f,P_n)$ and $L(f,P_n)$ if $f(x)=x$. The formula $1+2+3+\cdots+n=n(n+1)/2$ will be useful.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x)=x$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

Solution.

(a) In light of Theorem 7.2.8, it will suffice to show the equivalence of the following two statements.

(i) There exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0.$$

(ii) For every $\varepsilon>0$ there exists a partition P_ε of $[a,b]$ such that

$$U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon.$$

Suppose that (i) holds and let $\varepsilon>0$ be given. There exists an $N\in\mathbf{N}$ such that

$$|U(f,P_n)-L(f,P_n)|=U(f,P_n)-L(f,P_n)<\varepsilon$$

for all $n\geq N$. Hence we can take $P_\varepsilon=P_N$. Thus (ii) holds.

Suppose that (ii) holds. For each $n\in\mathbf{N}$ there exists a partition P_n of $[a,b]$ such that $U(f,P_n)-L(f,P_n)<\frac{1}{n}$, from which it is clear that

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0.$$

Now suppose that such a sequence of partitions exists, so that f is integrable on $[a,b]$. For each $n\in\mathbf{N}$ we have the inequalities

$$L(f,P_n)\leq L(f),\quad U(f)\leq U(f,P_n),\quad\text{and}\quad L(f,P_n)\leq U(f,P_n).$$

These imply that

$$L(f,P_n) - U(f,P_n) \le L(f) - U(f,P_n) = U(f) - U(f,P_n) \le U(f,P_n) - L(f,P_n).$$

The squeeze theorem then shows that $\lim_{n\to\infty} U(f,P_n) = U(f) = \int_a^b f$ and a similar argument shows that $\lim_{n\to\infty} L(f,P_n) = L(f) = \int_a^b f$.

(b) For each $0 \le k \le n-1$, let $x_k = \frac{k}{n-1}$, and let $P_n = \{x_0, x_1, ..., x_{n-1}\}$. Since f is strictly increasing on $[0,1]$, we then have

$$m_k = x_{k-1} = \frac{k-1}{n-1} \quad \text{and} \quad M_k = x_k = \frac{k}{n-1}$$

for each $1 \le k \le n-1$. It follows that

$$U(f,P_n) = \sum_{k=1}^{n-1} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n-1} \frac{k}{(n-1)^2} = \frac{n}{2(n-1)},$$

$$L(f,P_n) = \sum_{k=1}^{n-1} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n-1} \frac{k-1}{(n-1)^2} = \frac{n}{2(n-1)} - \frac{1}{n-1}.$$

(c) From part (b) we have

$$U(f,P_n) - L(f,P_n) = \frac{1}{n-1} \to 0.$$

It then follows from part (a) that f is integrable on $[0,1]$ and that

$$\int_0^1 f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{n}{2(n-1)} = \frac{1}{2}.$$

Exercise 7.2.4. Let g be bounded on $[a,b]$ and assume there exists a partition P with $L(g,P) = U(g,P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

Solution. Suppose $P = \{x_0, x_1, ..., x_n\}$ is such that $L(g,P) = U(g,P)$. Given that $m_k \le M_k$ for all $1 \le k \le n$, we have the implication

$$m_k < M_k \text{ for some } k \in \{1, ..., n\} \quad \Rightarrow \quad L(g,P) < U(g,P).$$

Since $L(g,P) \le U(g,P)$, the contrapositive of the implication above is

$$L(g,P) = U(g,P) \quad \Rightarrow \quad m_k = M_k \text{ for all } k \in \{1, ..., n\}.$$

Consider a subinterval $[x_{k-1}, x_k]$ for some $k \in \{1, ..., n\}$. Since $m_k = M_k$, it must be the case that g is constant on this subinterval, say $g(x) = c_k$ for all $x \in [x_{k-1}, x_k]$. In fact, since $g(x_k) = c_k = c_{k+1}$, we see that $c_1 = \cdots = c_n$. Denoting this common value by c, we then have $g(x) = c$ for all $x \in [a,b]$.

Since $U(g,P) - L(g,P) = 0$, an appeal to Theorem 7.2.8 shows that g is integrable. Let $S = U(g,P) = L(g,P)$. On one hand, $S = L(g,P)$ is a lower bound of the set $\{U(g,Q) : Q \in \mathcal{P})\}$, as we noted in Exercise 7.2.1. On the other hand, $S = U(g,P)$ belongs to the set $\{U(g,Q) : Q \in \mathcal{P})\}$ and hence must be the minimum of this set. Since the minimum and the infimum of a set necessarily coincide when they both exist, we see that

$$\int_a^b g = U(g) = U(g,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = c\sum_{k=1}^{n}(x_k - x_{k-1}) = c(x_n - x_0) = c(b-a).$$

Exercise 7.2.5. Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n) \to f$ uniformly on $[a,b]$, prove that f is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

Solution. Let $\varepsilon > 0$ be given. Because $f_n \to f$ uniformly, there exists an $N \in \mathbf{N}$ such that

$$n \geq N \text{ and } x \in [a,b] \quad \Rightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}. \tag{1}$$

By hypothesis the function f_N is integrable on $[a,b]$ and thus by Theorem 7.2.8 there exists a partition $P = \{x_0, ..., x_m\}$ of $[a,b]$ such that $U(f_N,P) - L(f_N,P) < \frac{\varepsilon}{3}$. Consider a subinterval $[x_{k-1}, x_k]$ for some $k \in \{1, ..., m\}$, and let

$$M_{k,N} = \sup\{f_N(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

Inequality (1) implies that

$$\left|M_{k,N} - M_k\right| \leq \frac{\varepsilon}{3(b-a)},$$

which gives us

$$|U(f_N,P) - U(f,P)| \leq \sum_{k=1}^{m}\left|M_{k,N} - M_k\right|(x_{k-1} - x_k) \leq \frac{\varepsilon}{3(b-a)}\sum_{k=1}^{m}(x_{k-1} - x_k) = \frac{\varepsilon}{3}.$$

We can similarly show that $|L(f_N,P) - L(f,P)| \leq \frac{\varepsilon}{3}$. It follows that

$$\begin{aligned} U(f,P) - L(f,P) \leq |U(f_N,P) - U(f,P)| + |L(f_N,P) - L(f,P)| \\ + |U(f_N,P) - L(f_N,P)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and an appeal to Theorem 7.2.8 allows us to conclude that f is integrable on $[a,b]$.

Exercise 7.2.6. A *tagged partition* $(P, \{c_k\})$ is one where in addition to a partition P we choose a sampling point c_k in each of the subintervals $[x_{k-1}, x_k]$. The corresponding *Riemann sum*,

$$R(f, P) = \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

is discussed in Section 7.1, where the following definition is alluded to.

Riemann's Original Definition of the Integral: A bounded function f is *integrable* on $[a, b]$ with $\int_a^b f = A$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta x_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \varepsilon.$$

Show that if f satisfies Riemann's definition above, then f is integrable in the sense of Definition 7.2.7. (The full equivalence of these two characterizations of integrability is proved in Section 8.1.)

Solution. Let $\varepsilon > 0$ be given. Since f satisfies Riemann's definition of integrability, there exists a $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta x_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \frac{\varepsilon}{2}.$$

Let $N \in \mathbf{N}$ be such that $\frac{b-a}{N} < \delta$, for each $k \in \{0, ..., N\}$ let $y_k = a + k\frac{b-a}{N}$, and let Q_1 be the partition $\{y_0, ..., y_N\}$ of $[a, b]$; note that $\Delta y_k = \frac{b-a}{N} < \delta$. Since $U(f)$ is the infimum of the set $\{U(f, Q) : Q \in \mathcal{P}\}$, there exists a partition Q_2 of $[a, b]$ such that

$$U(f) \leq U(f, Q_2) < U(f) + \frac{\varepsilon}{4}.$$

Let P be the common refinement of Q_1 and Q_2, say

$$P = Q_1 \cup Q_2 = \{x_0, ..., x_n\}.$$

Note that $\Delta x_k \leq \Delta y_k = \frac{b-a}{N} < \delta$, so that for any choice of sampling points we have

$$|R(f, P) - A| < \frac{\varepsilon}{2}. \tag{1}$$

Note further that since $Q_2 \subseteq P$, Lemma 7.2.3 gives us

$$U(f) \leq U(f, P) \leq U(f, Q_2) < U(f) + \frac{\varepsilon}{4}. \tag{2}$$

For each $k \in \{1, ..., n\}$, since M_k is the supremum of f over $[x_{k-1}, x_k]$, there exists some $c_k \in [x_{k-1}, x_k]$ such that

$$M_k - \frac{\varepsilon}{4(b-a)} < f(c_k) \leq M_k.$$

Take the collection $\{c_k\}$ as the sampling points for the partition P. It follows that

$$0 \leq U(f,P) - R(f,P) = \sum_{k=1}^{n}(M_k - f(c_k))\Delta x_k < \frac{\varepsilon}{4(b-a)} \sum_{k=1}^{n} \Delta x_k = \frac{\varepsilon}{4}. \tag{3}$$

Now observe that by (1), (2), and (3) we have

$$\begin{aligned} |U(f) - A| &\leq |U(f) - R(f,P)| + |R(f,P) - A| \\ &\leq |U(f) - U(f,P)| + |U(f,P) - R(f,P)| + |R(f,P) - A| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary it follows that $U(f) = A$. An analogous argument shows that $L(f) = A$ and thus $U(f) = L(f)$, i.e. f is integrable in the sense of Definition 7.2.7.

Exercise 7.2.7. Let $f : [a,b] \to \mathbf{R}$ be increasing on the set $[a,b]$ (i.e., $f(x) \leq f(y)$ whenever $x < y$). Show that f is integrable on $[a,b]$.

Solution. Let $\varepsilon > 0$ be given and let $n \in \mathbf{N}$ be such that

$$\frac{(b-a)(f(b) - f(a))}{n} < \varepsilon.$$

For $k \in \{0, ..., n\}$ let $x_k = a + k\frac{b-a}{n}$ and let P be the partition $\{x_0, ..., x_n\}$ of $[a,b]$. Note that, since f is increasing on $[a,b]$, we have

$$m_k = f(x_{k-1}) \quad \text{and} \quad M_k = f(x_k)$$

for each $k \in \{1, ..., n\}$. Hence

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) = \frac{(b-a)(f(b) - f(a))}{n} < \varepsilon \end{aligned}$$

and it follows from Theorem 7.2.8 that f is integrable on $[a,b]$.

7.3. Integrating Functions with Discontinuities

Exercise 7.3.1. Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 2 & \text{for } x = 1 \end{cases}$$

over the interval $[0, 1]$.

(a) Show that $L(f, P) = 1$ for every partition P of $[0, 1]$.

(b) Construct a partition P for which $U(f, P) < 1 + 1/10$.

(c) Given $\varepsilon > 0$, construct a partition P_ε for which $U(f, P_\varepsilon) < 1 + \varepsilon$.

Solution.

(a) Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0, 1]$. For any $1 \le k \le n$ we have

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = 1$$

and thus

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1 - 0 = 1.$$

(b) Let $x_0 = 0, x_1 = \frac{19}{20}, x_2 = 1$, and let P be the partition $\{x_0, x_1, x_2\}$ of $[0, 1]$. Observe that $M_1 = 1$ and $M_2 = 2$; it follows that

$$U(f, P) = M_1(x_1 - x_0) + M_2(x_2 - x_1) = 2 - x_1 = \frac{21}{20} = 1 + \frac{1}{20} < 1 + \frac{1}{10}.$$

(c) Let $x_0 = 0, x_1 = \max\{\frac{1}{2}, 1 - \frac{\varepsilon}{2}\}, x_2 = 1$, and let P be the partition $\{x_0, x_1, x_2\}$ of $[0, 1]$. Observe that $M_1 = 1$ and $M_2 = 2$; it follows that

$$U(f, P) = M_1(x_1 - x_0) + M_2(x_2 - x_1) = 2 - x_1 \le 1 + \frac{\varepsilon}{2} < 1 + \varepsilon.$$

Exercise 7.3.2. Recall that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has a countable set of discontinuities occurring at precisely every rational number. Follow these steps to prove $t(x)$ is integrable on $[0,1]$ with $\int_0^1 t = 0$.

(a) First argue that $L(t,P) = 0$ for any partition P of $[0,1]$.

(b) Let $\varepsilon > 0$, and consider the set of points $D_{\varepsilon/2} = \{x \in [0,1] : t(x) \geq \varepsilon/2\}$. How big is $D_{\varepsilon/2}$?

(c) To complete the argument, explain how to construct a partition P_ε of $[0,1]$ so that $U(t, P_\varepsilon) < \varepsilon$.

Solution.

(a) Let $P = \{x_0, x_1, ..., x_n\}$ be an arbitrary partition of $[0,1]$. The irrationals are dense in $\mathbf{R}$, so any subinterval $[x_{k-1}, x_k]$ contains an irrational number y. Since $t(y) = 0$ and $t(x) \geq 0$ for all $x \in [0,1]$, it follows that $m_k = 0$ and hence that $L(t,P) = 0$.

(b) Since $0 \leq t(x) \leq 1$ for all $x \in [0,1]$, if $\frac{\varepsilon}{2} > 1$ then $D_{\varepsilon/2}$ is empty. Suppose therefore that $0 < \frac{\varepsilon}{2} \leq 1$ and let N be the smallest positive integer such that $\frac{1}{N} < \frac{\varepsilon}{2}$. It follows that $D_{\varepsilon/2}$ consists precisely of those rational numbers $\frac{m}{n} \in [0,1]$ (in lowest terms with $n > 0$) with $1 \leq n \leq N$, of which there are only finitely many. Thus $D_{\varepsilon/2}$ is finite for any $\varepsilon > 0$.

(c) Let $\varepsilon > 0$ be given. If $D_{\varepsilon/2}$ is empty, i.e. if $0 \leq t(x) < \frac{\varepsilon}{2}$ for all $x \in [0,1]$, then let P_ε be the partition $\{0,1\}$ of $[0,1]$. For this partition we have

$$U(t, P_\varepsilon) = \sup\{t(x) : x \in [0,1]\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

Now suppose that $D_{\varepsilon/2}$ is not empty; by part (b) it must be the case that $D_{\varepsilon/2}$ is finite, say $D_{\varepsilon/2} = \{y_1, ..., y_m\}$ for some $m \in \mathbf{N}$ and some $y_1, ..., y_m \in [0,1]$. Let P_ε be the evenly spaced partition $\{x_0, ..., x_n\}$ of $[0,1]$ satisfying $\Delta x_k < \frac{\varepsilon}{2(m+1)}$ for each $k \in \{1, ..., n\}$. Decompose the set $\{1, ..., n\}$ into the disjoint union $A \cup A^c$, where

$$A = \big\{k \in \{1, ..., n\} : \text{there exists } j \in \{1, ..., m\} \text{ such that } y_j \in [x_{k-1}, x_k]\big\}.$$

Observe that

$$U(t, P_\varepsilon) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k \in A} M_k \Delta x_k + \sum_{k \notin A} M_k \Delta x_k. \tag{1}$$

Note that A can contain at most $m+1$ elements and also that $M_k \leq 1$ for any $k \in \{1, ..., n\}$. It follows that

$$\sum_{k \in A} M_k \Delta x_k < \sum_{k \in A} \frac{\varepsilon}{2(m+1)} \leq (m+1)\frac{\varepsilon}{2(m+1)} = \frac{\varepsilon}{2}. \tag{2}$$

Now suppose that $k \in \{1, ..., n\}$ is such that $k \notin A$, so that $f(x) < \frac{\varepsilon}{2}$ for all $x \in [x_{k-1}, x_k]$. Then $M_k \leq \frac{\varepsilon}{2}$ and it follows that

$$\sum_{k \notin A} M_k \Delta x_k \leq \frac{\varepsilon}{2} \sum_{k \notin A} \Delta x_k \leq \frac{\varepsilon}{2} \sum_{k=1}^{n} \Delta x_k = \frac{\varepsilon}{2}. \tag{3}$$

Combining (1), (2), and (3), we see that $U(t, P_\varepsilon) < \varepsilon$.

We have now shown that for any $\varepsilon > 0$ there exists a partition P_ε of $[0, 1]$ such that $U(t, P_\varepsilon) < \varepsilon$. From part (a) we have $L(t, P_\varepsilon) = 0$ and hence $U(t, P_\varepsilon) - L(t, P_\varepsilon) < \varepsilon$. Thus t is integrable on $[0, 1]$. Part (a) also shows that $\int_0^1 t = L(t) = 0$.

Exercise 7.3.3. Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution. Let $P = \{x_0, ..., x_n\}$ be an arbitrary partition of $[0, 1]$. The irrationals are dense in $\mathbf{R}$, so any subinterval $[x_{k-1}, x_k]$ contains an irrational number y. Since $f(y) = 0$ and $f(x) \geq 0$ for all $x \in [0, 1]$, it follows that $m_k = 0$ and hence that $L(t, P) = 0$. Because P was an arbitrary partition of $[0, 1]$, we have also shown that $L(f) = 0$; once we show that f is integrable on $[0, 1]$ it will follow that $\int_0^1 f = 0$.

Let $c \in (0, 1)$ be given and let N be the smallest natural number such that $\frac{1}{N+1} < c$. Observe that the restriction of f to $[c, 1]$ is the function

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, ..., \frac{1}{N}, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \in \mathbf{N}$, let $P_n = \{x_0, ..., x_n\}$ be the evenly spaced partition of $[c, 1]$ satisfying $\Delta x_k \leq \frac{1}{n}$. If we take n large enough so that $n \geq N$ and each of the points $1, \frac{1}{2}, ..., \frac{1}{N}$ belongs to exactly one of the subintervals $[x_{k-1}, x_k]$, then $M_k = 1$ for exactly N indices k and $M_k = 0$ otherwise; it follows that $U(f, P_n)$ eventually satisfies

$$U(f, P_n) = \sum_{k=1}^{n} M_k \Delta x_k \leq \frac{N}{n}.$$

Since $L(f, P_n) = 0$ by our previous discussion, the squeeze theorem gives us

$$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Thus, by Exercise 7.2.3, f is integrable on $[c, 1]$. Theorem 7.3.2 allows us to conclude that f is integrable on $[0, 1]$.

Exercise 7.3.4. Let f and g be functions defined on (possibly different) closed intervals, and assume the range of f is contained in the domain of g so that the composition $g \circ f$ is properly defined.

(a) Show, by example, that it is not the case that if f and g are integrable, then $g \circ f$ is integrable.

Now decide on the validity of each of the following conjectures, supplying a proof or counterexample as appropriate.

(b) If f is increasing and g is integrable, then $g \circ f$ is integrable.

(c) If f is integrable and g is increasing, then $g \circ f$ is integrable.

Solution.

(a) Let $f : [0, 1] \to \mathbf{R}$ be Thomae's function as defined in Exercise 7.3.2 as we showed there, f is integrable. Let $g : [0, 1] \to \mathbf{R}$ be given by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \leq 1. \end{cases}$$

Theorem 7.3.2 shows that g is also integrable. However, note that since $f(x) = 0$ for irrational x and $0 < f(x) \leq 1$ for rational x, the composition $g \circ f : [0, 1] \to \mathbf{R}$ is in fact Dirichlet's function, which was shown to be non-integrable in Example 7.3.3.

(b) This is false, however the only counterexample I know of is quite involved and uses material from Section 7.6.

(c) See part (a) for a counterexample.

Exercise 7.3.5. Provide an example or give a reason why the request is impossible.

(a) A sequence $(f_n) \to f$ pointwise, where each f_n has at most a finite number of discontinuities but f is not integrable.

(b) A sequence $(g_n) \to g$ uniformly where each g_n has at most a finite number of discontinuities and g is not integrable.

(c) A sequence $(h_n) \to h$ uniformly where each h_n is not integrable but h is integrable.

Solution.

(a) Define $f : [0, 1] \to \mathbf{R}$ and, for each $n \in \mathbf{N}$, $f_n : [0, 1] \to \mathbf{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \left[\frac{1}{n}, 1\right], \\ 0 & \text{if } x \in \left[0, \frac{1}{n}\right), \end{cases} \quad f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

Then $(f_n) \to f$ pointwise, each f_n has exactly one discontinuity at $x = \frac{1}{n}$, but f is not bounded and hence is not integrable.

(b) This is impossible. As discussed after Theorem 7.3.2, each g_n must be integrable. Exercise 7.2.5 then shows that g is integrable.

(c) For each $n \in \mathbf{N}$ define $h_n : [0,1] \to \mathbf{R}$ by

$$h_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

and let $h : [0,1] \to \mathbf{R}$ be identically zero. Then h is certainly integrable and a small modification of the argument given in Example 7.3.3 shows that each h_n is not integrable. Furthermore, since

$$\sup\{|h_n(x) - h(x)| : x \in [0,1]\} = \frac{1}{n} \to 0,$$

we have uniform convergence $(h_n) \to h$.

Exercise 7.3.6. Let $\{r_1, r_2, r_3, ...\}$ be an enumeration of all the rationals in $[0,1]$, and define

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is $G(x) = \sum_{n=1}^{\infty} g_n(x)$ integrable on $[0,1]$?

(b) Is $F(x) = \sum_{n=1}^{\infty} g_n(x)/n$ integrable on $[0,1]$?

Solution.

(a) For irrational $x \in [0,1]$ we have $g_n(x) = 0$ for all $n \in \mathbf{N}$ and thus $G(x) = 0$. If $x \in [0,1]$ is rational then $x = r_N$ for some $N \in \mathbf{N}$. Since $g_N(r_N) = 1$ and $g_n(r_N) = 0$ for $n \neq N$, we have $G(r_N) = 1$. Hence G is in fact Dirichlet's function, which is not integrable (Example 7.3.3).

(b) We claim that F is integrable on $[0,1]$; notice that

$$F(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

The density of the irrationals in $\mathbf{R}$ implies that $L(F,P) = 0$ for any partition P of $[0,1]$. Let $\varepsilon > 0$ be given and let $D = \left\{x \in [0,1] : F(x) \geq \frac{\varepsilon}{2}\right\}$. If $\frac{\varepsilon}{2} > 1$ then D is empty, since $0 \leq F(x) \leq 1$ for all $x \in [0,1]$. If $\frac{\varepsilon}{2} \leq 1$ then let N be the smallest positive integer such that $\frac{1}{N} < \frac{\varepsilon}{2}$; note that $N \geq 2$. It follows that $D = \{r_1, ..., r_{N-1}\}$, so that D is finite. We may now argue as in Exercise 7.3.2 (c) to obtain a partition P_ε of $[0,1]$ such that $U(F, P_\varepsilon) < \varepsilon$. Since $L(F, P_\varepsilon) = 0$ it follows that

$$U(F, P_\varepsilon) - L(F, P_\varepsilon) < \varepsilon.$$

Thus F is integrable on $[0,1]$. Furthermore, $\int_0^1 F = L(F) = 0$.

Exercise 7.3.7. Assume $f : [a, b] \to \mathbf{R}$ is integrable.

(a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a, b]$ then g is integrable as well.

(b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

Solution.

(a) Let $D = \{x \in [a, b] : f(x) \neq g(x)\}$. If D is empty then it is clear that g is integrable, so suppose that $D = \{c_1, ..., c_d\}$ for some $d \in \mathbf{N}$ and $c_1, ..., c_d \in [a, b]$. Let $\varepsilon > 0$ be given. Because f is integrable, there exists a partition Q_1 of $[a, b]$ such that

$$U(f, Q) - L(f, Q) < \varepsilon.$$

The integrability of f also implies that f is bounded; since g differs from f at only finitely many points, g must also be bounded, say by $R > 0$. Let Q_2 be the evenly spaced partition of $[a, b]$ whose subintervals have length less than $\frac{\varepsilon}{4R(d+1)}$ and let

$$P = Q_1 \cup Q_2 = \{x_0, ..., x_n\}$$

be the common refinement of Q_1 and Q_2. Note that $\Delta x_k < \frac{\varepsilon}{4R(d+1)}$. Let

$$M_k^g = \sup\{g(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad m_k^g = \inf\{g(x) : x \in [x_{k-1}, x_k]\}$$

and define M_k^f and m_k^f similarly. Decompose the set $\{1, ..., n\}$ into the disjoint union $A \cup A^c$, where

$$A = \{k \in \{1, ..., n\} : \text{there exists } j \in \{1, ..., d\} \text{ such that } c_k \in [x_{k-1}, x_k]\},$$

so that

$$U(g, P) - L(g, P) = \sum_{k \in A} (M_k^g - m_k^g)\Delta x_k + \sum_{k \notin A} (M_k^g - m_k^g)\Delta x_k. \tag{1}$$

Note that A can contain at most $d + 1$ elements and also that $M_k^g - m_k^g \leq 2R$ for any $k \in \{1, ..., n\}$. It follows that

$$\sum_{k \in A} (M_k^g - m_k^g)\Delta x_k < \sum_{k \in A} 2R\frac{\varepsilon}{4R(d+1)} \leq (d+1)\frac{\varepsilon}{2(d+1)} = \frac{\varepsilon}{2}. \tag{2}$$

Now suppose that $k \in \{1, ..., n\}$ is such that $k \notin A$, so that $f(x) = g(x)$ for all $x \in [x_{k-1}, x_k]$. It follows that $M_k^g - m_k^g = M_k^f - m_k^f$ and thus

$$\sum_{k \notin A} (M_k^g - m_k^g)\Delta x_k = \sum_{k \notin A} \left(M_k^f - m_k^f\right)\Delta x_k \leq \sum_{k=1}^{n} \left(M_k^f - m_k^f\right)\Delta x_k$$

$$= U(f, P) - L(f, P) \leq U(f, Q_1) - L(f, Q_1) < \frac{\varepsilon}{2}.$$

Combining this inequality with (1) and (2), we see that $U(g, P) - L(g, P) < \varepsilon$. Thus g is integrable on $[a, b]$.

(b) Let $f : [0,1] \to \mathbf{R}$ be identically zero, so that f is certainly integrable, and let $g : [0,1] \to \mathbf{R}$ be Dirichlet's function. Then g differs from f precisely on the countable set $\mathbf{Q} \cap [0,1]$ and yet g is not integrable.

Exercise 7.3.8. As in Exercise 7.3.6, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals in $[0,1]$, but this time define

$$h_n(x) = \begin{cases} 1 & \text{if } r_n < x \le 1 \\ 0 & \text{if } 0 \le x \le r_n. \end{cases}$$

Show $H(x) = \sum_{n=1}^{\infty} h_n(x)/2^n$ is integrable on $[0,1]$ even though it has discontinuities at every rational point.

Solution. For a given $N \in \mathbf{N}$ let $\sum_{n=1}^{N} h_n(x)/2^n$ and order the rationals $\{r_1, \ldots, r_N\}$ as $0 \le r_{i_1} < \cdots < r_{i_N} \le 1$. Then

$$H_N(x) = \begin{cases} 0 & \text{if } x \in [0, r_{i_1}], \\ \frac{1}{2} & \text{if } x \in (r_{i_1}, r_{i_2}], \\ \frac{3}{4} & \text{if } x \in (r_{i_2}, r_{i_3}], \\ \vdots & \qquad \vdots \\ 1 - 2^{-N} & \text{if } x \in (r_{i_N}, 1]. \end{cases}$$

Thus H_N is piecewise-constant on $[0,1]$. It is straightforward to argue that such functions are integrable (this is implied by Theorem 7.3.2 or Theorem 7.4.1). Now observe that

$$\left| \frac{h_n(x)}{2^n} \right| \le \frac{1}{2^n}$$

for each $n \in \mathbf{N}$. The Weierstrass M-Test (Corollary 6.4.5) now implies that H_N converges uniformly to H; it follows from Exercise 7.2.5 that H is integrable on $[0,1]$.

Exercise 7.3.9 (Content Zero). A set $A \subseteq [a, b]$ has *content zero* if for every $\varepsilon > 0$ there exists a finite collection of open intervals $\{O_1, O_2, ..., O_N\}$ that contain A in their union and whose lengths sum to ε or less. Using $|O_n|$ to refer to the length of each interval, we have

$$A \subseteq \bigcup_{n=1}^{N} O_n \quad \text{and} \quad \sum_{n=1}^{N} |O_n| \leq \varepsilon.$$

(a) Let f be bounded on $[a, b]$. Show that if the set of discontinuous points of f has content zero, then f is integrable.

(b) Show that any finite set has content zero.

(c) Content zero sets do not have to be finite. They do not have to be countable. Show that the Cantor set C defined in Section 3.1 has content zero.

(d) Prove that

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C, \end{cases}$$

is integrable, and find the value of the integral.

Solution.

(a) Suppose f is bounded by $R > 0$ on $[a, b]$ and let $\varepsilon > 0$ be given. Because the set of discontinuous points of f has content zero, we can choose a partition Q of $[a, b]$ such that the discontinuities of f are contained in the interiors of subintervals whose total length is strictly less than $\frac{\varepsilon}{4R}$. Letting K be the union of the remaining subintervals, we have that f is continuous on K and also that K is compact, being a finite union of closed and bounded intervals. Thus f is uniformly continuous on K and, as in the proof of Theorem 7.2.9, we may refine the partition Q, subdividing K as necessary, to obtain a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that the indices $\{1, ..., n\}$ can be expressed as the disjoint union $A \cup B$, where

(i) f is continuous on $\bigcup_{k \in A} [x_{k-1}, x_k]$ and $M_k - m_k < \frac{\varepsilon}{2(b-a)}$ for $k \in A$;

(ii) the discontinuities of f are contained inside $\bigcup_{k \in B} (x_{k-1}, x_k)$ and $\sum_{k \in B} \Delta x_k < \frac{\varepsilon}{4R}$.

It follows that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k) \Delta x_k \\ &= \sum_{k \in A} (M_k - m_k) \Delta x_k + \sum_{k \in B} (M_k - m_k) \Delta x_k \\ &< \frac{\varepsilon}{2(b-a)} \sum_{k \in A} \Delta x_k + 2R \sum_{k \in B} \Delta x_k \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus f is integrable on $[a, b]$.

(b) Let $A \subseteq \mathbf{R}$ be finite and let $\varepsilon > 0$ be given. If A is empty then the open interval $\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ suffices to show that A has content zero. Suppose therefore that A is not empty, say $A = \{x_1, ..., x_N\}$. For each $n \in \{1, ..., n\}$ let

$$O_n = \left(x_n - \frac{\varepsilon}{2N}, x_n + \frac{\varepsilon}{2N}\right).$$

Then $A \subseteq \bigcup_{n=1}^{N} O_n$ and

$$\sum_{n=1}^{N} |O_n| = \sum_{n=1}^{N} \frac{\varepsilon}{N} = \varepsilon.$$

Thus A has content zero.

(c) Recall from Section 3.1 that the Cantor set C is defined as the intersection $C = \bigcap_{n=0}^{\infty} C_n$, where each C_n consists of 2^n closed intervals each of length 3^{-n} and such that

$$\cdots \subseteq C_2 \subseteq C_1 \subseteq C_0 = [0, 1].$$

Let $\varepsilon > 0$ be given and choose $N \in \mathbf{N}$ such that

$$\left(\frac{2}{3}\right)^N + \left(\frac{1}{10}\right)^N < \varepsilon.$$

The set C_N consists of 2^N closed intervals each of length 3^{-N}; suppose there intervals are $[x_k, y_k]$ for $1 \leq k \leq 2^N$, so that $y_k - x_k = 3^{-N}$. For each $1 \leq k \leq 2^N$, let

$$O_k = \left(x_k - \frac{1}{2^{N+1} 10^N}, y_k + \frac{1}{2^{N+1} 10^N}\right),$$

so that $[x_k, y_k] \subseteq O_k$ and

$$|O_k| = \frac{1}{3^N} + \frac{1}{2^N 10^N}.$$

Now observe that

$$C = \bigcap_{n=0}^{\infty} C_n \subseteq C_N = \bigcup_{k=1}^{2^N} [x_k, y_k] \subseteq \bigcup_{k=1}^{2^N} O_k$$

$$\text{and} \quad \sum_{k=1}^{2^N} |O_k| = \sum_{k=1}^{2^N} \left(\frac{1}{3^N} + \frac{1}{2^N 10^N}\right) = \left(\frac{2}{3}\right)^N + \left(\frac{1}{10}\right)^N < \varepsilon.$$

Thus C has content zero.

(d) Let

$$D_h = \{x \in \mathbf{R} : h \text{ is not continuous at } x\}.$$

We claim that $D_h = C$. First, suppose that $x \notin C$. Since C is closed, the complement of C is open and thus there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq C^c$. Thus h is constant on the proper interval $(x-\delta, x+\delta)$; it follows that h is continuous at x. Now suppose that $x \in C$. To show that h is not continuous at x, it will suffice to show that for any $\delta > 0$ there exists some $y \in (x-\delta, x+\delta)$ such that $y \notin C$. The existence of some δ such that this does not hold implies that C contains a proper interval. However, C cannot contain any proper intervals since it is totally disconnected (Exercise 3.4.8). Thus h is not continuous at x and our claim follows.

Abbott does not specify an interval to integrate h over, but in fact h is integrable over any interval $[a, b]$ for $a < b$. Let $g : [a, b] \to \mathbf{R}$ be the restriction of h to $[a, b]$. Then

$$D_g = \{x \in [a, b] : g \text{ is not continuous at } x\} = D_h \cap [a, b] = C \cap [a, b].$$

It is straightforward to verify that if a set has content zero then the intersection of that set with any other set also has content zero. Thus, by part (c), D_g has content zero and it follows from part (a) that g is integrable. To calculate the integral of g, let P be any partition of $[a, b]$. As we noted before, C does not contain any proper intervals. It follows that any subinterval $[x_{k-1}, x_k]$ of the partition P contains some $x \notin C$, whence $g(x) = 0$. Thus $L(g, P) = 0$ and, because P was an arbitrary partition of $[a, b]$, it follows that

$$\int_a^b g = L(g) = 0.$$

7.4. Properties of the Integral

Exercise 7.4.1. Let f be a bounded function on a set A, and set

$$M = \sup\{f(x) : x \in A\}, \quad m = \inf\{f(x) : x \in A\},$$
$$M' = \sup\{|f(x)| : x \in A\}, \quad \text{and} \quad m' = \inf\{|f(x)| : x \in A\}.$$

(a) Show that $M - m \geq M' - m'$.

(b) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

(c) Provide the details for the argument that in this case we have $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Solution.

(a) Let $\varepsilon > 0$ be given. By Lemma 1.3.8 and Exercise 1.3.1 (b), there exist $x, y \in A$ such that

$$M' - \frac{\varepsilon}{2} < |f(x)| \quad \text{and} \quad |f(y)| < m' + \frac{\varepsilon}{2}.$$

It follows that

$$M' - m' - \varepsilon < |f(x)| - |f(y)| \leq \big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)| \leq M - m;$$

we have used the reverse triangle inequality (Exercise 1.2.6 (d)) for the third inequality. We have now shown that for all $\varepsilon > 0$ the inequality $M' - m' \leq M - m - \varepsilon$ holds and hence, by Exercise 1.2.10 (c), we may conclude that $M' - m' \leq M - m$.

(b) Let $\varepsilon > 0$ be given. Because f is integrable on $[a, b]$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. By part (a) we then have

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \varepsilon.$$

Thus $|f|$ is integrable on $[a, b]$.

(c) Since $f(x) \leq |f(x)|$ for all $x \in [a, b]$, Theorem 7.4.2 (iv) implies that

$$\int_a^b f \leq \int_a^b |f|. \tag{1}$$

Similarly, since $-f(x) \leq |f(x)|$ for all $x \in [a, b]$ we have $\int_a^b -f \leq \int_a^b |f|$ and it follows from Theorem 7.4.2 (ii) that

$$-\int_a^b f \leq \int_a^b |f|. \tag{2}$$

Combining (1) and (2), we see that $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Exercise 7.4.2.

(a) Let $g(x) = x^3$, and classify each of the following as positive, negative, or zero.

$$\text{(i)} \int_0^{-1} g + \int_0^1 g \qquad \text{(ii)} \int_1^0 g + \int_0^1 g \qquad \text{(iii)} \int_1^{-2} g + \int_0^1 g.$$

(b) Show that if $b \leq a \leq c$ and f is integrable on the interval $[b, c]$, then it is still the case that $\int_a^b f = \int_a^c f + \int_c^b f$.

Solution.

(a) For (i) we have, by Definition 7.4.3,

$$\int_0^{-1} g + \int_0^1 g = -\int_{-1}^0 g + \int_0^1 g.$$

By Theorem 7.4.1:

$$\int_0^1 g = \int_0^{1/2} g + \int_{1/2}^1 g.$$

As $g(x) \geq 0$ for all $x \in \left[0, \frac{1}{2}\right]$, Theorem 7.4.2 (iv) implies that $\int_0^{1/2} g \geq 0$. Similarly, since $g(x) \geq \frac{1}{8}$ for all $x \in \left[\frac{1}{2}, 1\right]$, we have by Theorem 7.4.2 (iv):

$$\int_{1/2}^1 g \geq \frac{1}{8}\left(1 - \frac{1}{2}\right) = \frac{1}{16} > 0.$$

It follows that $\int_0^1 g > 0$. By splitting the integral $\int_{-1}^0 g$ into $\int_{-1}^{-1/2} g + \int_{-1/2}^0 g$, we can similarly show that $\int_{-1}^0 g < 0$. We may conclude that

$$\int_0^{-1} g + \int_0^1 g = -\int_{-1}^0 g + \int_0^1 g > 0.$$

For (ii) we have, by Definition 7.4.3,

$$\int_1^0 g + \int_0^1 g = -\int_0^1 g + \int_0^1 g = 0.$$

For (iii) we have, by Definition 7.4.3 and Theorem 7.4.1,

$$\int_1^{-2} g + \int_0^1 g = -\int_{-2}^1 g + \int_0^1 g = -\left(\int_{-2}^0 + \int_0^1 g\right) + \int_0^1 g$$

$$= -\int_{-2}^0 g = -\left(\int_{-2}^{-1} g + \int_{-1}^0 g\right).$$

Because $g(x) \leq 0$ for all $x \in [-1, 0]$, Theorem 7.4.2 (iv) implies that $\int_{-1}^0 g \leq 0$. Similarly, since $g(x) \leq -1$ for all $x \in [-2, -1]$, Theorem 7.4.2 (iv) gives us $\int_{-2}^{-1} g \leq -1$. It follows that

$$\int_1^{-2} g + \int_0^1 g = -\left(\int_{-2}^{-1} g + \int_{-1}^0 g\right) \geq 1 > 0.$$

(b) By Theorem 7.4.1 we have

$$\int_b^c f = \int_b^a f + \int_a^c f.$$

By Definition 7.4.3 this is equivalent to

$$-\int_c^b f = -\int_a^b f + \int_a^c f,$$

which gives us

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Exercise 7.4.3. Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.

(b) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.

(c) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

Solution.

(a) This is false. For a counterexample, let $f : [0, 1] \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ -1 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Then $|f(x)| = 1$ for all $x \in [0, 1]$, so that $|f|$ is integrable on $[0, 1]$, but if f were integrable on $[0, 1]$ then $g(x) = \frac{1}{2}(f(x) + 1)$ would be integrable on $[0, 1]$ by Theorem 7.4.2 —but g is Dirichlet's function, which is non-integrable on $[0, 1]$ by Example 7.3.3.

(b) This is false. For a counterexample, see Exercise 7.3.3.

(c) This is true. Since g is continuous at y_0 there exists a $\delta > 0$ such that

$$g(x) \in (g(y_0) - \varepsilon, g(y_0) + \varepsilon)$$

for all $x \in I$, where $\varepsilon = g(y_0)/2 > 0$ and $I = [a, b] \cap (y_0 - \delta, y_0 + \delta)$. In particular,

$$g(x) > g(y_0) - \varepsilon = \varepsilon > 0 \quad \text{for all } x \in I.$$

Let $c = \inf I, d = \sup I$, and note that $0 < 2\delta \leq d - c \leq b - a$. By Theorem 7.4.1 we have

$$\int_a^b g = \int_a^c g + \int_c^d g + \int_d^b g.$$

Because g is non-negative, Theorem 7.4.2 (iv) implies that $\int_a^c g \geq 0$ and $\int_d^b g \geq 0$. Furthermore, Theorem 7.4.2 (iii) gives us

$$\int_c^d g \geq \varepsilon(d-c) > 0.$$

We may conclude that $\int_a^b g = \int_a^c g + \int_c^d g + \int_d^b g > 0$.

Exercise 7.4.4. Show that if $f(x) > 0$ for all $x \in [a,b]$ and f is integrable, then $\int_a^b f > 0$.

Solution. Let us first prove the following lemma.

Lemma L.17. Suppose $f : [a,b] \to \mathbf{R}$ is integrable and satisfies $\int_a^b f = 0$. Then for every $\varepsilon > 0$ there exists a closed and bounded interval $I \subseteq [a,b]$ such that $f(x) < \varepsilon$ for all $x \in I$.

Proof. Because $\int_a^b f = U(f) = 0$, there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that $0 \leq U(f,P) < \varepsilon(b-a)$. If $M_k \geq \varepsilon$ for all $k \in \{1, \ldots, n\}$ then

$$U(f,P) = \sum_{k=1}^{n} M_k \Delta x_k \geq \varepsilon \sum_{k=1}^{n} \Delta x_k = \varepsilon(b-a).$$

Given that $U(f,P) < \varepsilon(b-a)$, it must be the case that there is some $k \in \{1, \ldots, n\}$ such that $M_k < \varepsilon$. The desired interval is then $I = [x_{k-1}, x_k]$. $\square$

Now let us return to the exercise. It is immediate from Theorem 7.4.2 (iv) that $\int_a^b f \geq 0$. Suppose that $\int_a^b f = 0$; we will show that this leads to a contradiction. By Lemma L.17, there exists a closed and bounded interval $I_1 \subseteq [a,b]$ such that $f(x) < 1$ for all $x \in I_1$. Theorem 7.4.1 shows that f is integrable on I_1. Furthermore, since f is positive and $\int_a^b f = 0$, the integral of f over I_1 must also be zero. Lemma L.17 then implies that there is some closed and bounded interval $I_2 \subseteq I_1$ such that $f(x) < \frac{1}{2}$ for all $x \in I_2$. Continuing in this manner, we obtain a nested sequence of closed and bounded intervals

$$\cdots \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq [a,b]$$

such that if $x \in I_n$ then $f(x) < \frac{1}{n}$. The Nested Interval Property (Theorem 1.4.1) implies that the intersection $\bigcap_{n=1}^{\infty} I_n$ is non-empty, so that there exists some $x_0 \in I_n$ for each $n \in \mathbf{N}$, which implies that $f(x_0) < \frac{1}{n}$ for all $n \in \mathbf{N}$. It follows that $f(x_0) \leq 0$, contradicting the positivity of f. We may conclude that $\int_a^b f > 0$.

Exercise 7.4.5. Let f and g be integrable functions on $[a, b]$.

(a) Show that if P is any partition of $[a, b]$, then

$$U(f+g, P) \leq U(f, P) + U(g, P).$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

(b) Review the proof of Theorem 7.4.2 (ii), and provide an argument for part (i) of this theorem.

Solution.

(a) Let $P = \{x_0, ..., x_n\}$ be a partition of $[a, b]$ and, for each $k \in \{1, ..., n\}$, let

$$M_k^f = \sup\{f(x) : x \in [x_{k-1}, x_k]\};$$

define M_k^g and M_k^{f+g} similarly. Let $k \in \{1, ..., n\}$ be given. For any $\varepsilon > 0$, Lemma 1.3.8 implies that there is some $x \in [x_{k-1}, x_k]$ such that

$$M_k^{f+g} - \varepsilon < f(x) + g(x) \leq M_k^f + M_k^g.$$

So for any $\varepsilon > 0$ we have $M_k^{f+g} \leq M_k^f + M_k^g + \varepsilon$; Exercise 1.2.10 (c) allows us to conclude that $M_k^{f+g} \leq M_k^f + M_k^g$. It follows that

$$U(f+g, P) \leq U(f, P) + U(g, P).$$

For an example where this inequality is strict, let $f, g : [0, 1] \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{if } 0 < x < 1, \\ 3 & \text{if } x = 1, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3 & \text{if } x = 0, \\ 2 & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 1, \end{cases}$$

so that

$$f(x) + g(x) = \begin{cases} 3 & \text{if } x = 0 \text{ or } x = 1, \\ 4 & \text{if } 0 < x < 1. \end{cases}$$

For the partition $P = \{0, 1\}$ of $[0, 1]$ we then have

$$U(f+g, P) = \sup\{f(x) + g(x) : x \in [0, 1]\} = 4,$$

$$U(f, P) = \sup\{f(x) : x \in [0, 1]\} = 3, \quad \text{and} \quad U(g, P) = \sup\{g(x) : x \in [0, 1]\} = 3.$$

Thus $U(f+g, P) = 4 < 6 = U(f, P) + U(g, P)$.

The corresponding inequality for lower sums is

$$L(f, P) + L(g, P) \leq L(f+g, P),$$

which can be proved similarly; we can also find an analogous example showing that this inequality can be strict.

(b) Because f and g are integrable on $[a,b]$, Exercise 7.2.3 implies that there are sequences (Q_n) and (R_n) of partitions of $[a,b]$ such that

$$\lim_{n\to\infty}[U(f,Q_n)-L(f,Q_n)] = \lim_{n\to\infty}[U(g,R_n)-L(g,R_n)] = 0.$$

For each $n \in \mathbf{N}$ let $P_n = Q_n \cup R_n$ be the common refinement of Q_n and R_n. Lemma 7.2.3 then gives us the inequalities

$$0 \le U(f,P_n)-L(f,P_n) \le U(f,Q_n)-L(f,Q_n)$$

$$\text{and} \quad 0 \le U(g,P_n)-L(g,P_n) \le U(g,R_n)-L(g,R_n);$$

together with the Squeeze Theorem, these imply that

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)] = \lim_{n\to\infty}[U(g,P_n)-L(g,P_n)] = 0.$$

By part (a) we have the inequality

$$0 \le U(f+g,P_n)-L(f+g,P_n) \le U(f,P_n)-L(f,P_n)+U(g,P_n)-L(g,P_n)$$

and so another application of the Squeeze Theorem gives us

$$\lim_{n\to\infty}[U(f+g,P_n)-L(f+g,P_n)] = 0.$$

Exercise 7.2.3 then implies that $f+g$ is integrable on $[a,b]$ and also that

$$\int_a^b (f+g) = \lim_{n\to\infty} U(f+g,P_n) = \lim_{n\to\infty} L(f+g,P_n).$$

Again by part (a) and Exercise 7.2.3 we have

$$\int_a^b (f+g) = \lim_{n\to\infty} U(f+g,P_n) \le \lim_{n\to\infty}[U(f,P_n)+U(g,P_n)] = \int_a^b f + \int_a^b g.$$

Similarly,

$$\int_a^b f + \int_a^b g = \lim_{n\to\infty}[L(f,P_n)+L(g,P_n)] \le \lim_{n\to\infty} L(f+g,P_n) = \int_a^b (f+g).$$

We may conclude that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Exercise 7.4.6. Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact:

(a) If f satisfies $|f(x)| \leq M$ on $[a, b]$, show
$$\left|(f(x))^2 + (f(y))^2\right| \leq 2M|f(x) - f(y)|.$$

(b) Prove that if f is integrable on $[a, b]$, then so is f^2.

(c) Now show that if f and g are integrable, then fg is integrable. (Consider $(f + g)^2$.)

Solution.

(a) For any $x, y \in [a, b]$ we have
$$\begin{aligned}\left|(f(x))^2 - (f(y))^2\right| &= |f(x) + f(y)||f(x) - f(y)| \\ &\leq (|f(x)| + |f(y)|)|f(x) - f(y)| \leq 2M|f(x) - f(y)|.\end{aligned}$$

(b) Because f is integrable on $[a, b]$ it is bounded on $[a, b]$, say by $R > 0$. Suppose $P = \{x_0, \ldots, x_n\}$ is an arbitrary partition of $[a, b]$. For $k \in \{1, \ldots, n\}$ define
$$M(k, f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad m(k, f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\};$$
define $M(k, f^2)$ and $m(k, f^2)$ similarly. Let $k \in \{1, \ldots, n\}$ and $\delta > 0$ be given. By Lemma 1.3.8 and Exercise 1.3.1 (b), there exist $x, y \in [x_{k-1}, x_k]$ such that
$$M(k, f^2) - \frac{\delta}{2} < (f(x))^2 \quad \text{and} \quad (f(y))^2 < m(k, f^2) + \frac{\delta}{2}.$$
Together with part (a) these inequalities give us
$$\begin{aligned}M(k, f^2) - m(k, f^2) < (f(x))^2 - (f(y))^2 &\leq \left|(f(x))^2 - (f(y))^2\right| \\ &\leq 2R|f(x) - f(y)| \leq 2R(M(k, f) - m(k, f)).\end{aligned}$$
We have now shown that $M(k, f^2) - m(k, f^2) \leq 2R(M(k, f) - m(k, f)) + \delta$ for all $\delta > 0$. It follows from Exercise 1.2.10 (c) that
$$M(k, f^2) - m(k, f^2) \leq 2R(M(k, f) - m(k, f)),$$
which implies that $U(f^2, P) - L(f^2, P) \leq 2R[U(f, P) - L(f, P)]$.

Now let $\varepsilon > 0$ be given. Since f is integrable on $[a, b]$ there exists a partition P of $[a, b]$ such that
$$U(f, P) - L(f, P) < \frac{\varepsilon}{2R}.$$
By our previous discussion, it follows that
$$U(f^2, P) - L(f^2, P) \leq 2R[U(f, P) - L(f, P)] < \varepsilon.$$
Thus f^2 is integrable on $[a, b]$.

(c) Since $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$, it follows from part (b), Theorem 7.4.2 (i), and Theorem 7.4.2 (ii) that fg is integrable on $[a,b]$.

Exercise 7.4.7. Review the discussion immediately preceding Theorem 7.4.4.

(a) Produce an example of a sequence $f_n \to 0$ pointwise on $[0,1]$ where $\lim_{n\to\infty} \int_0^1 f_n$ does not exist.

(b) Produce an example of a sequence g_n with $\int_0^1 g_n \to 0$ but $g_n(x)$ does not converge to zero for any $x \in [0,1]$. To make it more interesting, let's insist that $g_n(x) \geq 0$ for all x and n.

Solution.

(a) Let (f_n) be the sequence given by

$$f_n(x) = \begin{cases} (-1)^n n & \text{if } 0 < x < \frac{1}{n}, \\ 0 & \text{if } x = 0 \text{ or } \frac{1}{n} \leq x \leq 1. \end{cases}$$

Then $f_n \to 0$ pointwise on $[0,1]$, but

$$\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} (-1)^n$$

does not exist.

(b) For subsets $A \subseteq B \subseteq \mathbf{R}$, denote by $\chi_A : B \to \mathbf{R}$ the indicator/characteristic function of A within B, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Define a sequence of functions $(g_n : [0,1] \to \mathbf{R})$ by

$$g_1 = \chi_{[0,1]},$$

$$g_2 = \chi_{[0,\frac{1}{2}]}, g_3 = \chi_{[\frac{1}{2},1]},$$

$$g_4 = \chi_{[0,\frac{1}{3}]}, g_5 = \chi_{[\frac{1}{3},\frac{2}{3}]}, g_6 = \chi_{[\frac{2}{3},1]},$$

and so on; this sequence is sometimes called the typewriter sequence (see here for a variant of this sequence). Observe that

$$\int_0^1 g_1 = 1, \quad \int_0^1 g_2 = \int_0^1 g_3 = \frac{1}{2}, \quad \int_0^1 g_4 = \int_0^1 g_5 = \int_0^1 g_6 = \frac{1}{3}, \quad \text{etc.},$$

so that $\int_0^1 g_n \to 0$. However, for any $x \in [0,1]$ and any $N \in \mathbf{N}$, there always exists some natural number $n \geq N$ such that $g_n(x) = 1$; it follows that $(g_n(x))$ does not converge to zero.

Exercise 7.4.8. For each $n \in \mathbf{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 1/2^n \end{cases},$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show H is integrable and compute $\int_0^1 H$.

Solution. For each $N \in \mathbf{N}$, let $H_N : [0,1] \to \mathbf{R}$ be the N^{th} partial sum of H. Observe that

$$H_N(x) = \begin{cases} 0 & \text{if } x \in \left[0, \dfrac{1}{2^N}\right], \\ \dfrac{2^k - 1}{2^N} & \text{if } x \in \left(\dfrac{1}{2^{N-k+1}}, \dfrac{1}{2^{N-k}}\right], 1 \leq k \leq N. \end{cases}$$

Thus each H_N is piecewise-constant. Theorem 7.4.1 then implies that each H_N is integrable and furthermore that

$$\int_0^1 H_N = \sum_{k=1}^{N} \int_{2^{-(N-k+1)}}^{2^{-(N-k)}} H_N = \sum_{k=1}^{N} \left(\frac{2^k - 1}{2^N}\right)\left(\frac{1}{2^{N-k}} - \frac{1}{2^{N-k+1}}\right).$$

Some calculations reveal that

$$\int_0^1 H_N = \frac{2}{3} - \frac{1}{6 \cdot 4^{N-1}} - \frac{1}{4^N} + \frac{1}{2^{N+1}},$$

so that $\lim_{N\to\infty} \int_0^1 H_N = \frac{2}{3}$. The Weierstrass M-Test implies that H_N converges uniformly to H on $[0,1]$. It follows from Theorem 7.4.4 that H is integrable on $[0,1]$ and also that

$$\int_0^1 H = \lim_{N\to\infty} \int_0^1 H_N = \frac{2}{3}.$$

Exercise 7.4.9. Let g_n and g be uniformly bounded on $[0,1]$, meaning that there exists a single $M > 0$ satisfying $|g(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in [0,1]$. Assume $g_n \to g$ pointwise on $[0,1]$ and uniformly on any set of the form $[0,\alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Solution. Let $\varepsilon > 0$ be given and let $\alpha = \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon}{4M}\right\}$. Because $g_n \to g$ uniformly on $[0,\alpha]$, there exists an $N \in \mathbf{N}$ such that

$$x \in [0,\alpha] \text{ and } n \geq N \quad \Rightarrow \quad |g_n(x) - g(x)| < \frac{\varepsilon}{2\alpha}.$$

Then, provided $n \geq N$, we have

$$
\begin{aligned}
\left|\int_0^1 g_n(x)\,\mathrm{d}x - \int_0^1 g(x)\,\mathrm{d}x\right| &= \left|\int_0^1 g_n(x) - g(x)\,\mathrm{d}x\right| \\
&\le \int_0^1 |g_n(x) - g(x)|\,\mathrm{d}x \\
&= \int_0^\alpha |g_n(x) - g(x)|\,\mathrm{d}x + \int_\alpha^1 |g_n(x) - g(x)|\,\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + 2M(1-\alpha) \\
&\le \varepsilon,
\end{aligned}
$$

where we have used Theorem 7.4.2 for the first, second, and fourth lines, and Theorem 7.4.1 for the third line. It follows that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Exercise 7.4.10. Assume g is integrable on $[0,1]$ and continuous at 0. Show

$$
\lim_{n\to\infty} \int_0^1 g(x^n)\,dx = g(0).
$$

Solution. Since g is integrable on $[0,1]$ it is bounded on $[0,1]$, say by $M > 0$. Let $\varepsilon > 0$ be given and let $\alpha = \max\{\frac{1}{2}, 1 - \frac{\varepsilon}{4M}\}$. Because g is continuous at 0 there exists a $\delta > 0$ such that

$$
x \in [0,\delta) \cap [0,1] \quad \Rightarrow \quad |g(x) - g(0)| < \frac{\varepsilon}{2\alpha}.
$$

Since $\frac{1}{2} \le \alpha < 1$, we have $\lim_{n\to\infty} \alpha^n = 0$. Thus there exists an $N \in \mathbf{N}$ such that

$$
n \ge N \quad \Rightarrow \quad 0 \le \alpha^n < \delta.
$$

Suppose $n \ge N$ and $x \in [0,\alpha]$. Then by the previous discussion we have

$$
0 \le x^n \le \alpha^n < \delta \quad \Rightarrow \quad |g(x^n) - g(0)| < \frac{\varepsilon}{2\alpha}.
$$

It follows that for $n \ge N$ we have

$$
\begin{aligned}
\left|\int_0^1 g(x^n)\,\mathrm{d}x - g(0)\right| &= \left|\int_0^1 g(x^n) - g(0)\,\mathrm{d}x\right| \\
&\le \int_0^1 |g(x^n) - g(0)|\,\mathrm{d}x \\
&= \int_0^\alpha |g(x^n) - g(0)|\,\mathrm{d}x + \int_\alpha^1 |g(x^n) - g(0)|\,\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + 2M(1-\alpha) \\
&= \varepsilon,
\end{aligned}
$$

where the first, second, and fourth lines follow from Theorem 7.4.2 and the third line follows from Theorem 7.4.1. Thus $\lim_{n\to\infty} \int_0^1 g(x^n)\ \mathrm{d}x = g(0)$.

Exercise 7.4.11. Review the original definition of integrability in Section 7.2, and in particular the definition of the upper integral $U(f)$. One reasonable suggestion might be to bypass the complications introduced in Definition 7.2.7 and simply define the integral to be the value of $U(f)$. Then *every* bounded function is integrable! Although tempting, proceeding in this way has some significant drawbacks. Show by example that several of the properties in Theorem 7.4.2 no longer hold if we replace our current definition of integrability with the proposal that $\int_a^b f = U(f)$ for every bounded function f.

Solution. We will consider each of the properties in Theorem 7.4.2.

(i) This property no longer holds. For example, consider $f, g : [0,1] \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x \in \mathbf{Q}, \\ 1 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

so that $(f+g)(x) = 1$ for all $x \in [0,1]$. In this case we have

$$U(f) = U(g) = U(f+g) = 1$$

and thus $U(f+g) \neq U(f) + U(g)$.

(ii) This property no longer holds. For example, take f to be Dirichlet's function on $[0,1]$. Then $U(-f) = 0 \neq -1 = -U(f)$.

(iii) This property still holds, and follows as in the textbook, i.e. by observing that

$$L(f,P) \leq U(f) \leq U(f,P)$$

for any partition P and then taking P to be the partition $\{a, b\}$.

(iv) This property still holds. Let $P = \{x_0, ..., x_n\}$ be a partition of $[a,b]$ and note that the inequality $f(x) \leq g(x)$ for all $x \in [a,b]$ implies that

$$\sup\{f(x) : x \in [x_{k-1}, x_k]\} \leq \sup\{g(x) : x \in [x_{k-1}, x_k]\}$$

for any $k \in \{1, ..., n\}$. It follows that $U(f,P) \leq U(g,P)$. Since P was an arbitrary partition, we may conclude that $U(f) \leq U(g)$.

(v) This property still holds. Since f is bounded if and only if $|f|$ is bounded, $|f|$ is "integrable" (in the sense of this exercise). The inequality $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a,b]$, combined with property (iv), gives us the inequalities $U(f) \leq U(|f|)$ and $U(-f) \leq U(|f|)$. Now, for any $\varepsilon > 0$, there exists a partition P such that

$$\begin{aligned} &U(-f) \leq U(-f,P) < U(-f) + \varepsilon \\ &\qquad \Rightarrow \quad -U(-f) - \varepsilon < -U(-f,P) = L(f,P) \leq L(f) \leq U(f). \end{aligned}$$

It follows that $-U(-f) \leq U(f)$, which gives us $-U(f) \leq U(-f)$ and hence, by our previous discussion, $-U(f) \leq U(|f|)$. This inequality, together with the inequality $U(f) \leq U(|f|)$, allows us to conclude that $|U(f)| \leq U(|f|)$.

7.5. The Fundamental Theorem of Calculus

Exercise 7.5.1.

(a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Solution.

(a) Some calculations reveal that $F : [-1, \infty) \to \mathbf{R}$ is given by

$$F(x) = \begin{cases} \frac{1}{2}(1 - x^2) & \text{if} -1 \leq x \leq 0, \\ \frac{1}{2}(1 + x^2) & \text{if } x > 0. \end{cases}$$

It is straightforward to manually check that F is differentiable (and hence continuous) on its domain, with derivative given by $F'(x) = f(x)$. However, note that the Fundamental Theorem of Calculus part (ii) (FToC, Theorem 7.5.1 (ii)) immediately implies that F is continuous on any interval of the form $[-1, b]$ for $b \in \mathbf{R}$ (in fact, Lipschitz on such intervals) and hence is continuous on its domain. Furthermore, as f is continuous everywhere, the FToC also implies that F is differentiable on its domain with derivative given by $F'(x) = f(x)$.

(b) In this case, the function $F : [-1, \infty) \to \mathbf{R}$ is given by

$$F(x) = \begin{cases} 1 + x & \text{if} -1 \leq x \leq 0, \\ 1 + 2x & \text{if } x > 0. \end{cases}$$

As in part (a), the FToC part (ii) implies that F is continuous on its domain. Furthermore, since f is continuous on $A = [-1, 0) \cup (0, \infty)$, the FToC implies that F is differentiable on A with derivative given by $F'(x) = f(x)$. However, since f is not continuous at 0 the FToC does not allow us to conclude that F is differentiable at 0. Indeed, F fails to be differentiable here:

$$\lim_{x \to 0^-} \frac{F(x) - F(0)}{x} = 1 \neq 2 = \lim_{x \to 0^+} \frac{F(x) - F(0)}{x}.$$

Exercise 7.5.2. Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $g = h'$ for some h on $[a, b]$, then g is continuous on $[a, b]$.

(b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.

(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c.

Solution.

(a) This is false. For a counterexample, consider the function $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} x^{5/3} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then, as we showed in Exercise 5.2.7 (a), h is differentiable but h' is not continuous at 0.

(b) This is true. Theorem 7.2.9 implies that g is integrable on $[a, b]$ and so we are justified in defining $h : [a, b] \to \mathbf{R}$ by $h(x) = \int_a^x g$. The continuity of g on $[a, b]$ then allows us to use the FToC part (ii) to conclude that $g = h'$.

(c) This is false. For a counterexample, consider $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $H : [-1, 1] \to \mathbf{R}$ defined by $H(x) = \int_{-1}^x h(t)\, dt$ is identically zero and hence differentiable at 0, but h is not continuous at 0.

Exercise 7.5.3. The hypothesis in Theorem 7.5.1 (i) that $F'(x) = f(x)$ for all $x \in [a, b]$ is slightly stronger than it needs to be. Carefully read the proof and state exactly what needs to be assumed with regard to the relationship between f and F for the proof to be valid.

Solution. In light of Theorem 7.4.1 and the fact that the Mean Value Theorem only requires differentiability on an open interval, it would suffice for F to be continuous on $[a, b]$ and $F'(x) = f(x)$ to hold for all but finitely many $x \in [a, b]$.

Exercise 7.5.4. Show that if $f : [a, b] \to \mathbf{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Solution. Define $F : [a, b] \to \mathbf{R}$ by $F(x) = \int_a^x f$. On one hand, since by assumption F is identically zero on $[a, b]$, we have that F is differentiable on $[a, b]$ and satisfies $F'(x) = 0$ for all $x \in [a, b]$. On the other hand, because f is continuous on $[a, b]$, the FToC part (ii) implies that $F'(x) = f(x)$ for all $x \in [a, b]$. Thus f is identically zero on $[a, b]$.

For an example demonstrating that this conclusion does not follow if f is not continuous, consider $f : [0, 1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Then $\int_0^x f = 0$ for all $x \in [0, 1]$, but f is not identically zero.

Exercise 7.5.5. The Fundamental Theorem of Calculus can be used to supply a shorter argument for Theorem 6.3.1 under the additional assumption that the sequence of derivatives is continuous.

Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a, b]$. Assuming each f_n' is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a, b]$. Show that $g(x) = f'(x)$.

Solution. Let $x \in [a, b]$ be given. Because $f_n' \to g$ uniformly on $[a, x]$, Theorem 7.4.4 shows that

$$\lim_{n\to\infty} \int_a^x f_n' = \int_a^x g.$$

We can then take the limit as $n \to \infty$ on both sides of the equation $\int_a^x f_n' = f_n(x) - f_n(a)$ and use the pointwise convergence $f_n \to f$ to see that

$$f(x) = f(a) + \int_a^x g$$

for all $x \in [a, b]$. Since g is the uniform limit of a sequence of continuous functions it is itself continuous (Theorem 6.2.6) and so we may invoke the FToC part (ii) to conclude that $f'(x) = g(x)$ for all $x \in [a, b]$.

Exercise 7.5.6 (Integration-by-parts).

(a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt.$$

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

Solution.

(a) By assumption the functions h, h', k, and k' are continuous on $[a, b]$; it follows that $(hk)' = hk' + h'k$ is continuous on $[a, b]$. Theorem 7.2.9 then implies that $(hk)'$ is integrable on $[a, b]$ and so we may use the FToC part (i) to see that

$$\int_a^b h(t)k'(t) + h'(t)k(t)\ \mathrm{d}t = \int_a^b (h(t)k(t))'\ \mathrm{d}t = h(b)k(b) - h(a)k(a).$$

(b) In light of Exercise 7.4.6, we need only assume that h' and k' are integrable on $[a, b]$.

Exercise 7.5.7. Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that f is continuous. (To get started, set $G(x) = \int_a^x f$.)

Solution. It will suffice to show that $G(b) = F(b) - F(a)$. Because f is continuous on $[a, b]$, the FToC part (ii) implies that $G'(x) = f(x) = F'(x)$ for all $x \in [a, b]$. It follows from Corollary 5.3.4 that $G(x) = F(x) + k$ for some constant k. Substituting $x = a$, we see that $k = -F(a)$ and thus $G(b) = F(b) - F(a)$, as desired.

Exercise 7.5.8 (Natural Logarithm and Euler's Constant). Let

$$L(x) = \int_1^x \frac{1}{t}\, dt,$$

where we consider only $x > 0$.

(a) What is $L(1)$? Explain why L is differentiable and find $L'(x)$.

(b) Show that $L(xy) = L(x) + L(y)$. (Think of y as a constant and differentiate $g(x) = L(xy)$.)

(c) Show $L(x/y) = L(x) - L(y)$.

(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.

(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Solution.

(a) We have $L(1) = 0$. Because t^{-1} is continuous on $(0, \infty)$, the FToC part (ii) shows that L is differentiable on $(0, \infty)$ and satisfies $L'(x) = x^{-1}$.

(b) Note that, by part (a),

$$\frac{\mathrm{d}}{\mathrm{d}x}L(xy) = yL'(xy) = \frac{y}{xy} = \frac{1}{x} = L'(x).$$

Corollary 5.3.4 then implies that $L(xy) = L(x) + k$ for some constant k. Substituting $x = 1$ we see that $k = L(y)$ and thus $L(xy) = L(x) + L(y)$, as desired.

(c) Observe that, by parts (a) and (b),

$$0 = L(1) = L\left(\frac{y}{y}\right) = L(y) + L\left(\frac{1}{y}\right),$$

so that $L(1/y) = -L(y)$ for any $y > 0$. Combining this with part (b) shows that $L(x/y) = L(x) - L(y)$.

(d) Let $n \geq 2$ be given and consider the partition $P = \{1, ..., n\}$ of $[1, n]$. Observe that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} = U\left(\frac{1}{t}, P\right) \geq U\left(\frac{1}{t}\right) = L(n).$$

Thus $\gamma_n \geq 0$ for each $n \in \mathbf{N}$, so that (γ_n) is bounded below.

Now let $n \in \mathbf{N}$ be given and observe that

$$\gamma_n - \gamma_{n+1} = L\left(1 + \frac{1}{n}\right) - \frac{1}{n+1}.$$

Since $t^{-1} \geq n(n+1)^{-1}$ on $\left[1, 1 + n^{-1}\right]$, Theorem 7.42 (iii) shows that

$$L\left(1 + \frac{1}{n}\right) \geq \frac{1}{n+1}$$

and hence $\gamma_n \geq \gamma_{n+1}$ for each $n \in \mathbf{N}$, so that (γ_n) is decreasing. We can now appeal to the Monotone Convergence Theorem (Theorem 2.4.2) to conclude that (γ_n) converges.

(e) For $n \in \mathbf{N}$, observe that

$$\begin{aligned}
\gamma_{2n} - \gamma_n &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - L(2n) + L(n) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(\frac{2}{2} + \frac{2}{4} + \cdots + \frac{2}{2n}\right) - L(2) - L(n) + L(n) \\
&= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right) - L(2).
\end{aligned}$$

Taking the limit as $n \to \infty$ on both sides gives the desired equality.

Exercise 7.5.9. Given a function f on $[a, b]$, define the *total variation* of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\},$$

where the supremum is taken over all partitions P of $[a, b]$.

(a) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \le \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

Solution.

(a) Let $P = \{x_0, ..., x_n\}$ be an arbitrary partition of $[a, b]$. Because f' is continuous on $[a, b]$ it is integrable on $[a, b]$ and so we may use the FToC part (i) and Theorem 7.4.2 (v) to see that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|.$$

As P was arbitrary, it follows that $Vf \le \int_a^b |f'|$.

(b) For any $\varepsilon > 0$ there exists a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that

$$\left(\int_a^b |f'|\right) - \varepsilon = L(|f'|) - \varepsilon < L(|f'|, P).$$

For any $k \in \{1, ..., n\}$, apply the Mean Value Theorem on the interval $[x_{k-1}, x_k]$ to obtain some $t_k \in (x_{k-1}, x_k)$ such that

$$|f'(t_k)|(x_k - x_{k-1}) = |f(x_k) - f(x_{k-1})|.$$

It follows that

$$\begin{aligned} L(|f'|, P) &= \sum_{k=1}^{n} \inf\{|f'(t)| : t \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) \\ &\le \sum_{k=1}^{n} |f'(t_k)|(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ &\le Vf. \end{aligned}$$

We have now shown that for every $\varepsilon > 0$,

$$\int_a^b |f'| \le Vf + \varepsilon$$

and thus we obtain the inequality $\int_a^b |f'| \le Vf$. Given part (a), we may conclude that $Vf = \int_a^b |f'|$.

Exercise 7.5.10 (Change-of-variable Formula). Let $g : [a,b] \to \mathbf{R}$ be differentiable and assume g' is continuous. Let $f : [c,d] \to \mathbf{R}$ be continuous, and assume that the range of g is contained in $[c,d]$ so that the composition $f \circ g$ is properly defined.

(a) Why are we sure f is the derivative of some function? How about $(f \circ g)g'$?

(b) Prove the change-of-variable formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt.$$

Solution.

(a) Since f is continuous on $[c,d]$ it is integrable on $[c,d]$ and thus, letting $F(x) = \int_c^x f$, the FToC part (ii) gives us $F'(x) = f(x)$ for each $x \in [c,d]$. Similarly, note that $f \circ g$ is continuous on $[a,b]$, being a composition of continuous functions, and hence is integrable on $[a,b]$. By assumption g' is continuous on $[a,b]$ and thus g' is integrable on $[a,b]$. We can now use Exercise 7.4.6 to see that $(f \circ g)g'$ is integrable on $[a,b]$. Thus, letting $G(x) = \int_a^x (f \circ g)g'$, the FToC part (ii) shows that $G'(x) = f(g(x))g'(x)$ for each $x \in [a,b]$.

(b) Define $F : [c,d] \to \mathbf{R}$ and $G : [a,b] \to \mathbf{R}$ by

$$F(t) = \int_{g(a)}^t f(x)\ \mathrm{d}x \quad \text{and} \quad G(t) = \int_a^t f(g(x))g'(x)\ \mathrm{d}x.$$

Then $F'(t) = f(t)$, so that $[F(g(t))]' = f(g(t))g'(t)$, and $G'(t) = f(g(t))g'(t)$. It follows that $F(g(t)) = G(t) + k$ on $[a,b]$ for some constant k. Substituting $t = a$, we see that $k = 0$ and thus $F(g(b)) = G(b)$, i.e.

$$\int_{g(a)}^{g(b)} f(x)\ \mathrm{d}x = \int_a^b f(g(x))g'(x)\ \mathrm{d}x.$$

Exercise 7.5.11. Assume f is integrable on $[a,b]$ and has a "jump discontinuity" at $c \in (a,b)$. This means that both one-sided limits exist as x approaches c from the left and from the right, but that

$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x).$$

(This phenomenon is discussed in more detail in Section 4.6.)

(a) Show that, in this case, $F(x) = \int_a^x f$ is not differentiable at $x = c$.

(b) The discussion in Section 5.5 mentions the existence of a continuous monotone function that fails to be differentiable on a dense subset of $\mathbf{R}$. Combine the results of part (a) with Exercise 6.4.10 to show how to construct such a function.

Solution.

(a) Let $A = \lim_{x\to c^-} f(x)$ and $B = \lim_{x\to c^+} f(x)$. A small modification of the proof of the FToC part (ii) shows that

$$\lim_{x\to c^-} \frac{F(x)-F(c)}{x-c} = A \quad \text{and} \quad \lim_{x\to c^+} \frac{F(x)-F(c)}{x-c} = B.$$

Since $A \neq B$, we see that $\lim_{x\to c} \frac{F(x)-F(c)}{x-c}$ does not exist, i.e. F is not differentiable at c.

(b) As in Exercise 6.4.10, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals and for each $n \in \mathbf{N}$ define $u_n : \mathbf{R} \to \mathbf{R}$ by

$$u_n(x) = \begin{cases} 2^{-n} & \text{if } r_n < x, \\ 0 & \text{if } x \leq r_n. \end{cases}$$

Now define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = \sum_{n=1}^\infty u_n(x)$. Let $[a,b]$ be a given interval and note that for each $N \in \mathbf{N}$ the partial sum function $h_N(x) = \sum_{n=1}^N u_n(x)$ has at most N discontinuities on $[a,b]$. It follows from Theorem 7.4.1 that h_N is integrable on $[a,b]$. In Exercise 6.4.10 we showed that $h_N \to h$ uniformly on $\mathbf{R}$ and hence by Theorem 7.4.4 we see that h is integrable on $[a,b]$. We can now define $H : \mathbf{R} \to \mathbf{R}$ by $H(x) = \int_0^x h$. The FToC part (ii) shows that H is continuous, and we can use Theorem 7.4.1 and the fact that h is non-negative to see that H is monotone increasing.

Now we will prove that h has a jump discontinuity at each rational number. Let $r_m \in \mathbf{Q}$ be given; we have two claims.

(i) Our first claim is that $\lim_{x\to r_m^-} h(x) = h(r_m)$. To see this, let $\varepsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $2^{-N} < \varepsilon$. Because the set $\{r_1, \ldots, r_N\}$ is finite, we can choose a $\delta > 0$ such that the intersection $(r_m - \delta, r_m) \cap \{r_1, \ldots, r_N\}$ is empty, i.e. if $r_n \in (r_m - \delta, r_m)$, then $n > N$.

Now suppose that $x \in (r_m - \delta, r_m)$ and enumerate the rationals in $[x, r_m)$ as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$ of the sequence $\{r_1, r_2, r_3, \ldots\}$; by our previous discussion we must have $n_k > N$ for each $k \in \mathbf{N}$. As we showed in Exercise 6.4.10, h is strictly increasing and $h(r_m) - h(x) = \sum_{k=1}^{\infty} 2^{-n_k}$. Thus

$$|h(r_m) - h(x)| = 2^{-N} \sum_{k=1}^{\infty} 2^{-n_k+N} \leq 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \varepsilon$$

and our claim follows.

(ii) Our second claim is that $\lim_{x \to r_m^+} h(x) = h(r_m) + 2^{-m}$. Again, let $\varepsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $2^{-N} < \varepsilon$. Similarly to (i), we can choose a $\delta > 0$ such that if $r_n \in (r_m, r_m + \delta)$ then $n > N$. For $x \in (r_m, r_m + \delta)$, enumerate the rationals in (r_m, x) as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, ...\}$ of the sequence $\{r_1, r_2, r_3, ...\}$, so that

$$[r_m, x) = \{r_m, r_{n_1}, r_{n_2}, r_{n_3}, ...\};$$

by our previous discussion, we must have $n_k > N$ for each $k \in \mathbf{N}$. Thus

$$h(x) - h(r_m) = 2^{-m} + \sum_{k=1}^{n} 2^{-n_k}.$$

Arguing as in (i), it follows that

$$|h(x) - h(r_m) - 2^{-m}| = \sum_{k=1}^{\infty} 2^{-n_k} \leq 2^{-N} < \varepsilon.$$

This proves our second claim.

We have now shown that if $r_m \in \mathbf{Q}$ then

$$\lim_{x \to r_m^-} h(x) = h(r_m) < h(r_m) + 2^{-m} = \lim_{x \to r_m^+} h(x),$$

so that h has a jump discontinuity at each rational number. It follows from part (a) that H fails to be differentiable at each rational number.

7.6. Lebesgue's Criterion for Riemann Integrability

Exercise 7.6.1.

(a) First, argue that $L(t, P) = 0$ for any partition P of $[0, 1]$.

(b) Consider the set of points $D_{\varepsilon/2} = \{x : t(x) \geq \varepsilon/2\}$. How big is $D_{\varepsilon/2}$?

(c) To complete the argument, explain how to construct a partition P_ε of $[0, 1]$ so that $U(t, P_\varepsilon) < \varepsilon$.

Solution. See Exercise 7.3.2.

Exercise 7.6.2. Define

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}.$$

(a) Show h has discontinuities at each point of C and is continuous at every point of the complement of C. Thus, h is not continuous on an uncountably infinite set.

(b) Now prove that h is integrable on $[0, 1]$.

Solution. See Exercise 7.3.9.

Exercise 7.6.3. Show that any countable set has measure zero.

Solution. Let $A \subseteq \mathbf{R}$ be a countable set, i.e. $A = \{a_1, a_2, a_3, ...\}$, and let $\varepsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $2^{-N} < \varepsilon$. For each $n \in \mathbf{N}$, let

$$O_n = \left(a_n - \frac{\varepsilon}{2^{N+n+1}}, a_n + \frac{\varepsilon}{2^{N+n+1}}\right).$$

Then $A \subseteq \cup_{n=1}^{\infty} O_n$ and $|O_n| = 2^{-N-n}$, so that

$$\sum_{n=1}^{\infty} |O_n| = \sum_{n=1}^{\infty} 2^{-N-n} = 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \varepsilon.$$

Thus A has measure zero.

Exercise 7.6.4. Prove that the Cantor set has measure zero.

Solution. See Exercise 7.3.9.

Exercise 7.6.5. Show that if two sets A and B each have measure zero, then $A \cup B$ has measure zero as well. In addition, discuss the proof of the stronger statement that the countable union of sets of measure zero also has measure zero. (This second statement is true, but a completely rigorous proof requires a result about double summations discussed in Section 2.8.)

Solution. Let $\varepsilon > 0$ be given. Because A and B have measure zero, there are countable collections $\{O_1, O_2, O_3, ...\}$ and $\{U_1, U_2, U_3, ...\}$ of open intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} O_n, \quad \sum_{n=1}^{\infty} |O_n| \leq \frac{\varepsilon}{2}, \quad B = \bigcup_{n=1}^{\infty} U_n, \quad \text{and} \quad \sum_{n=1}^{\infty} |U_n| \leq \frac{\varepsilon}{2}.$$

By Theorem 1.5.8 (i) the union $\{O_1, O_2, O_3, ...\} \cup \{U_1, U_2, U_3, ...\}$ is a countable collection of open intervals, say

$$\{O_1, O_2, O_3, ...\} \cup \{U_1, U_2, U_3, ...\} = \{V_1, V_2, V_3, ...\}.$$

Is is then immediate that

$$A \cup B \subseteq \left(\bigcup_{n=1}^{\infty} O_n \right) \cup \left(\bigcup_{n=1}^{\infty} U_n \right) = \bigcup_{n=1}^{\infty} V_n.$$

Now, for any $N \in \mathbf{N}$ we can express the set $\{V_1, ..., V_N\}$ as a disjoint union $\mathbf{O} \cup \mathbf{U}$, where

$$\mathbf{O} \subsetneq \{O_1, O_2, O_3, ...\} \quad \text{and} \quad \mathbf{U} \subsetneq \{U_1, U_2, U_3, ...\};$$

$\mathbf{O}$ and $\mathbf{U}$ are both finite and either (but not both) of them can be empty. The decomposition $\{V_1, ..., V_N\} = \mathbf{O} \cup \mathbf{U}$ implies that

$$\sum_{n=1}^{N} |V_N| = \sum_{O \in \mathbf{O}} |O| + \sum_{U \in \mathbf{U}} |U| \leq \sum_{n=1}^{\infty} |O_n| + \sum_{n=1}^{\infty} |U_n| \leq \varepsilon.$$

Since N was arbitrary, we see that the sum $\sum_{n=1}^{\infty} |V_n|$ is convergent and satisfies $\sum_{n=1}^{\infty} |V_n| \leq \varepsilon$. Thus $A \cup B$ has measure zero.

Now suppose that $\{A_m : m \in \mathbf{N}\}$ is a countable collection of sets of measure zero; we will show that $\bigcup_{m=1}^{\infty} A_m$ also has measure zero. Let $\varepsilon > 0$ and $m \in \mathbf{N}$ be given. Because A_m has measure zero, there is a countable collection $\{O_{m,1}, O_{m,2}, O_{m,3}, ...\}$ of open intervals such that

$$A_m \subseteq \bigcup_{n=1}^{\infty} O_{m,n} \quad \text{and} \quad \sum_{n=1}^{\infty} |O_{m,n}| \leq 2^{-m} \varepsilon.$$

By Theorem 1.5.8 (ii), the union $\bigcup_{m=1}^{\infty} \{O_{m,1}, O_{m,2}, O_{m,3}, ...\}$ is a countable collection of open intervals, say

$$\bigcup_{m=1}^{\infty} \{O_{m,1}, O_{m,2}, O_{m,3}, ...\} = \{U_1, U_2, U_3, ...\}.$$

It is straightforward to verify that

$$\bigcup_{m=1}^{\infty} A_m \subseteq \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} O_{m,n} = \bigcup_{m=1}^{\infty} U_m.$$

Now let $M \in \mathbf{N}$ be given and consider the collection $\{U_1, ..., U_M\}$. Each U_k in this collection is equal to $O_{m,n}$ for some positive integers m and n. If we let K be the maximum of these positive integers m and n, then because each $|O_{m,n}|$ is non-negative we have the inequality

$$\sum_{m=1}^{M} |U_m| \leq \sum_{m=1}^{K} \sum_{n=1}^{K} |O_{m,n}|. \tag{1}$$

Keeping in mind that all the terms $|O_{m,n}|$ are non-negative, by assumption the sum $\sum_{n=1}^{\infty} |O_{m,n}|$ is convergent for each fixed $m \in \mathbf{N}$ and satisfies $\sum_{n=1}^{\infty} |O_{m,n}| \leq 2^{-m}\varepsilon$; by comparison we see that the iterated sum $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |O_{m,n}|$ converges and satisfies

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |O_{m,n}| \leq \sum_{m=1}^{\infty} 2^{-m}\varepsilon = \varepsilon. \tag{2}$$

We can now use (1), (2), and Theorem 2.8.1 to see that

$$\sum_{m=1}^{M} |U_m| \leq \sum_{m=1}^{K} \sum_{n=1}^{K} |O_{m,n}| \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |O_{m,n}| \leq \varepsilon.$$

Thus the sum $\sum_{m=1}^{\infty} |U_m|$ is convergent and does not exceed ε. We may conclude that $\bigcup_{m=1}^{\infty} A_m$ has measure zero.

Exercise 7.6.6. If $\alpha < \alpha'$, show that $D^{\alpha'} \subseteq D^{\alpha}$.

Solution. See Exercise 4.6.9.

Exercise 7.6.7.

(a) Let $\alpha > 0$ be given. Show that if f is continuous at $x \in [a, b]$, then it is α-continuous at x as well. Explain how it follows that $D^{\alpha} \subseteq D$.

(b) Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now, explain why this guarantees that

$$D = \bigcup_{n=1}^{\infty} D^{\alpha_n} \quad \text{where } \alpha_n = 1/n.$$

Solution.

(a) See Exercise 4.6.10.

(b) See Exercise 4.6.11.

Exercise 7.6.8. Prove that for a fixed $\alpha > 0$, the set D^α is closed.

Solution. See Exercise 4.6.8.

Exercise 7.6.9. Show that there exists a *finite* collection of disjoint open intervals

$$\{G_1, G_2, ..., G_N\}$$

whose union contains D^α and that satisfies

$$\sum_{n=1}^{N} |G_n| < \frac{\varepsilon}{4M}.$$

Solution. Because D has measure zero, there exists a countable collection $\{U_n : n \in \mathbf{N}\}$ of open intervals such that

$$D \subseteq \bigcup_{n=1}^{\infty} U_n \quad \text{and} \quad \sum_{n=1}^{\infty} |U_n| < \frac{\varepsilon}{4M}.$$

Observe that:

(i) D^α is closed by Exercise 7.6.8;

(ii) D^α is bounded since $D^\alpha \subseteq [a, b]$;

(iii) $D^\alpha \subseteq D \subseteq \bigcup_{n=1}^{\infty} U_n$ by Exercise 7.6.7 (a).

It follows from Theorem 7.6.4 that there is a finite subcollection $\{G_1, ..., G_N\}$ of $\{U_n : n \in \mathbf{N}\}$ such that $D^\alpha \subseteq \bigcup_{n=1}^{N} G_n$. Note that, for $1 \leq i < j \leq N$, if the intersection $G_i \cap G_j$ is non-empty then the union $G_i \cup G_j$ is also an open interval. Thus, by replacing G_i and G_j with their union if necessary, we can assume that the collection $\{G_1, ..., G_N\}$ is pairwise-disjoint (although it may no longer be a subset of $\{U_n : n \in \mathbf{N}\}$ after this replacement process; this is not important for the proof). Because each G_i originally came from the collection $\{U_n : n \in \mathbf{N}\}$, and the replacement process described previously will not increase the total length of the intervals (since $|G_i \cup G_j| \leq |G_i| + |G_j|$), we must have the inequality

$$\sum_{n=1}^{N} |G_n| \leq \sum_{n=1}^{\infty} |U_n| < \frac{\varepsilon}{4M}.$$

Exercise 7.6.10. Let K be what remains of the interval $[a, b]$ after the open intervals G_n are all removed; that is, $K = [a, b] \setminus \bigcup_{n=1}^{N} G_n$. Argue that f is uniformly α-continuous on K.

Solution. Since D^α is contained in the union $\bigcup_{n=1}^{N} G_n$, it must be the case that f is α-continuous on K, and since K is compact (being closed and bounded), it follows from the discussion after Exercise 7.6.8 in the textbook that f is uniformly α-continuous on K.

Exercise 7.6.11. Finish the proof in this direction by explaining how to construct a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \varepsilon$. It will be helpful to break the sum

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$$

into two parts—one over those subintervals that contain points of D^α and the other over subintervals that do not.

Solution. Since f is uniformly α-continuous on K (Exercise 7.6.10), there exists a $\delta > 0$ such that

$$x, y \in K \quad \text{and} \quad |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \alpha. \tag{1}$$

Notice that K is a finite union of closed and bounded intervals. We can subdivide these intervals to obtain a partition $\{t_0, ..., t_m\}$ of K such that $\Delta t_k < \delta$. Suppose that $G_j = (y_j, z_j)$ and define the following partition of $[a, b]$:

$$P_\varepsilon = \{t_0, ..., t_m, y_1, z_1, y_2, z_2, ..., y_N, z_N\} = \{x_0, x_1, ..., x_n\};$$

here we are relabeling so that the set $\{x_0, x_1, ..., x_n\}$ is ordered, i.e. $x_0 < x_1 < \cdots < x_n$. Now decompose the indices $\{1, ..., n\}$ into the disjoint union $A \cup A^c$, where

$$A = \{k \in \{1, ..., n\} : [x_{k-1}, x_k] \cap D^\alpha \neq \emptyset\},$$

i.e. A consists of those indices k such that the interval $[x_{k-1}, x_k]$ contains points of D^α. In other words, from the construction of P_ε, we have $(x_{k-1}, x_k) = G_j$ for some j. It follows from Exercise 7.6.9 that $\sum_{k \in A} \Delta x_k < \frac{\varepsilon}{4M}$. Because f is bounded by M on $[a, b]$ we then have

$$\sum_{k \in A} (M_k - m_k)\Delta x_k \leq 2M \sum_{k \in A} \Delta x_k < \frac{\varepsilon}{2}. \tag{2}$$

Now observe that the union $\bigcup_{k \notin A} [x_{k-1}, x_k]$ is the set K from Exercise 7.6.10, so that for $k \notin A$ we have $\Delta x_k = \Delta t_j < \delta$ for some j. It follows from (1) that $M_k - m_k \leq \alpha$ and thus

$$\sum_{k \notin A} (M_k - m_k)\Delta x_k \leq \alpha \sum_{k \notin A} \Delta x_k \leq \alpha \sum_{k=1}^{n} \Delta x_k = \frac{\varepsilon}{2(b - a)}(b - a) = \frac{\varepsilon}{2}. \tag{3}$$

Combining (2) and (3), we see that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k = \sum_{k \in A} (M_k - m_k)\Delta x_k + \sum_{k \notin A} (M_k - m_k)\Delta x_k < \varepsilon.$$

Exercise 7.6.12.

(a) Prove that D^α has measure zero. Point out that it is possible to choose a cover for D^α that consists of a finite number of open intervals.

(b) Show how this implies that D has measure zero.

Solution.

(a) If D^α is finite then we are done. Otherwise, suppose $P_\varepsilon = \{x_0, ..., x_n\}$ and let

$$A = \{k \in \{1, ..., n\} : (x_{k-1}, x_k) \cap D^\alpha \neq \emptyset\};$$

note that A must be non-empty since D^α is not finite. For $k \in A$, there exists some $x \in (x_{k-1}, x_k)$ such that f is not α-continuous at x. It follows that there exist points y and z in (x_{k-1}, x_k) such that $|f(y) - f(z)| \geq \alpha$, which implies that $M_k - m_k \geq \alpha$. Given this, it must be the case that $\sum_{k\in A} \Delta x_k < \varepsilon$. Indeed, if this were not the case then

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \geq \sum_{k\in A} (M_k - m_k)\Delta x_k \geq \alpha\varepsilon.$$

Thus $\{(x_{k-1}, x_k) : k \in A\}$ is a finite collection of open intervals whose total length is strictly less than ε.

Now observe that the union $\bigcup_{k\in A}(x_{k-1}, x_k)$ covers all but finitely many of the points of D^α; it may fail to cover the elements of P_ε, if any of these belong to D^α. Letting $E = P_\varepsilon \cap D^\alpha$, we then have

$$D^\alpha \subseteq E \cup \bigcup_{k\in A} (x_{k-1}, x_k).$$

If E is empty then we are done, since $\{(x_{k-1}, x_k) : k \in A\}$ is finite,

$$D^\alpha \subseteq \bigcup_{k\in A} (x_{k-1}, x_k), \quad \text{and} \quad \sum_{k\in A} \Delta x_k < \varepsilon.$$

Otherwise, suppose $E = \{x_{k_1}, ..., x_{k_m}\}$. Define

$$r = \frac{\varepsilon - \sum_{k\in A} \Delta x_k}{2m} \quad \text{and} \quad U_j = \left(x_{k_j} - \tfrac{r}{2}, x_{k_j} + \tfrac{r}{2}\right).$$

Then

$$\sum_{j=1}^{m} |U_j| = \sum_{j=1}^{m} r = \frac{\varepsilon - \sum_{k\in A} \Delta x_k}{2} \quad \Rightarrow \quad \sum_{k\in A} \Delta x_k + \sum_{j=1}^{m} |U_j| = \frac{\varepsilon + \sum_{k\in A} \Delta x_k}{2} < \varepsilon.$$

Thus $\{(x_{k-1}, x_k) : k \in A\} \cup \{U_1, ..., U_m\}$ is a finite collection of open intervals whose union contains D^α and whose total length is strictly less than ε. We may conclude that D^α has measure zero.

(b) By Exercise 7.6.7 (b), we may express D as the countable union

$$D = \bigcup_{n=1}^{\infty} D^{1/n};$$

by part (a) each $D^{1/n}$ has measure zero and so we may use Exercise 7.6.5 to conclude that D has measure zero.

Exercise 7.6.13.

(a) Show that if f and g are integrable on $[a, b]$, then so is the product fg. (This result was requested in Exercise 7.4.6, but notice how much easier the argument is now.)

(b) Show that if g is integrable on $[a, b]$ and f is continuous on the range of g, then the composition $f \circ g$ is integrable on $[a, b]$.

Solution.

(a) Let D_f be the set of discontinuities of f; define D_g and D_{fg} similarly. The contrapositive of Theorem 4.3.4 (iii) shows that $D_{fg} \subseteq D_f \cup D_g$. Because f and g are integrable on $[a, b]$, Lebesgue's Theorem (Theorem 7.6.5) shows that D_f and D_g have measure zero and it then follows from Exercise 7.6.5 that $D_f \cup D_g$ has measure zero. It is straightforward to verify that any subset of a measure zero set also has measure zero and thus D_{fg} has measure zero. Lebesgue's Theorem allows us to conclude that fg is integrable on $[a, b]$.

(b) Let D_g be the set of discontinuities of g and $D_{f\circ g}$ similarly. Given that f is continuous on the range of g, the contrapositive of Theorem 4.3.9 shows that $D_{f\circ g} \subseteq D_g$. Because g is integrable on $[a, b]$, Lebesgue's Theorem (Theorem 7.6.5) shows that D_g has measure zero and it follows that $D_{f\circ g}$ has measure zero; Lebesgue's Theorem allows us to conclude that $f \circ g$ is integrable on $[a, b]$.

Exercise 7.6.14.

(a) Find $g'(0)$.

(b) Use the standard rules of differentiation to compute $g'(x)$ for $x \neq 0$.

(c) Explain why, for every $\delta > 0, g'(x)$ attains every value between 1 and -1 as x ranges over the set $(-\delta, \delta)$. Conclude that g' is not continuous at $x = 0$.

Solution.

(a) The Squeeze Theorem shows that

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

(b) The standard rules of differentiation give us

$$g'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

for $x \neq 0$.

(c) For $n \in \mathbf{N}$ define

$$x_n = \frac{1}{2\pi n + \pi} \quad \text{and} \quad y_n = \frac{1}{2\pi n}.$$

Notice that:

(i) $\lim_{n\to\infty} y_n = 0$;
(ii) $0 < x_n < y_n$;
(iii) $g'(x_n) = 1$;
(iv) $g'(y_n) = -1$.

Let $\delta > 0$ be given. By (i) there exists an $N \in \mathbf{N}$ such that $y_N < \delta$; combined with (ii), we see that $x_N, y_N \in (-\delta, \delta)$. It now follows from (iii), (iv), and Darboux's Theorem (Theorem 5.2.7) that g' attains every value in $[-1, 1]$ on the interval $[x_N, y_N] \subseteq (-\delta, \delta)$. Because $\delta > 0$ was arbitrary, we see that g' cannot be continuous at 0.

Exercise 7.6.15.

(a) If $c \in C$, what is $\lim_{n\to\infty} f_n(c)$?

(b) Why does $\lim_{n\to\infty} f_n(x)$ exist for $x \notin C$?

Solution.

(a) Since f_n vanishes on C_n, and hence on C, for each $n \in \mathbf{N}$, we see that $\lim_{n\to\infty} f_n(c) = 0$.

(b) If $x \notin C$, then $x \in C_N^{\mathrm{c}}$ for some $N \in \mathbf{N}$. The sequence (f_n) is constructed so that

$$f_N(y) = f_{N+1}(y) = f_{N+2}(y) = \cdots$$

for all $y \in C_N^{\mathrm{c}}$. Thus the sequence $(f_n(x))$ is eventually constant and hence convergent.

Exercise 7.6.16.

(a) Explain why $f'(x)$ exists for all $x \notin C$.

(b) If $c \in C$, argue that $|f(x)| \leq (x - c)^2$ for all $x \in [0, 1]$. Show how this implies $f'(c) = 0$.

(c) Give a careful argument for why $f'(x)$ fails to be continuous on C. Remember that C contains many points besides the endpoints of the intervals that make up $C_1, C_2, C_3, \ldots$.

Solution.

(a) If $x \notin C$ then $x \in C_N^{\mathrm{c}}$ for some $N \in \mathbf{N}$. The sequence (f_n) is constructed so that

$$f_N(y) = f_{N+1}(y) = f_{N+2}(y) = \cdots$$

for all $y \in C_N^c$. Because f is the pointwise limit of (f_n), we see that $f(y) = f_N(y)$ for all $y \in C_N^c$. Since C_N^c is open, there exists some open interval U containing x and contained inside C_N^c, so that f and f_N agree on U; the differentiability of f_N on U then implies that $f'(x) = f_N'(x)$.

(b) As we showed in Exercise 3.4.3, there is a sequence (x_n), where each x_n is an endpoint of one of the intervals making up C_n, such that $\lim_{n\to\infty} x_n = c$. Let $x \in [0,1]$ be given. The sequence (f_n) is constructed so that

$$|f_n(x)| \leq (x - x_n)^2.$$

Taking the limit as $n \to \infty$ on both sides of this inequality gives us $|f(x)| \leq (x-c)^2$.

Now observe that, since $f(c) = 0$ (by Exercise 7.6.15 (a)),

$$\left|\frac{f(x) - f(c)}{x - c}\right| = \frac{|f(x)|}{|x - c|} \leq |x - c|.$$

It follows from the Squeeze Theorem that

$$\lim_{x\to c}\left|\frac{f(x) - f(c)}{x - c}\right| = 0,$$

which implies that $f'(c) = 0$.

(c) Suppose $x \in [0,1]$ is an endpoint of one of the intervals making up some C_n. We constructed f so that its behaviour near x is the same as the behaviour of g near 0. Thus, by a similar argument to the one given in Exercise 7.6.14 (c), for each $\delta > 0$ the derivative f' attains every value between 1 and -1 on the interval $(x - \delta, x + \delta)$.

Now, as we showed in Exercise 3.4.3, there is a sequence (x_n), where each x_n is an endpoint of one of the intervals making up C_n, such that $\lim_{n\to\infty} x_n = c$. Let $\delta > 0$ be given. There is an $N \in \mathbf{N}$ such that $x_N \in \left(c - \frac{\delta}{2}, c + \frac{\delta}{2}\right)$, which implies that

$$\left(x_N - \frac{\delta}{2}, x_N + \frac{\delta}{2}\right) \subseteq (c - \delta, c + \delta).$$

As we noted in the previous paragraph, f' must attain every value between 1 and -1 on the interval $\left(x_N - \frac{\delta}{2}, x_N + \frac{\delta}{2}\right)$ and hence on the interval $(c - \delta, c + \delta)$. As $\delta > 0$ was arbitrary, we see that f' is not continuous at c.

Exercise 7.6.17. Why is $f'(x)$ Riemann-integrable on $[0,1]$?

Solution. Suppose $x \notin C$. As we showed in Exercise 7.6.16 (a), there exists some open interval U containing x and some $N \in \mathbf{N}$ such that f and f_N agree on U. Since f_N is continuously differentiable on U, it follows that f' is continuous at x. Combined with Exercise 7.6.16 (c), this shows that the set of discontinuities of f' is precisely C, which has measure zero by Exercise 7.6.4. Lebesgue's Theorem now implies that f' is integrable on $[0,1]$.

Exercise 7.6.18. Show that, under these circumstances, the sum of the lengths of the intervals making up each C_n no longer tends to zero as $n \to \infty$. What is this limit?

Solution. The sum of the lengths of the intervals being removed is now

$$\sum_{n=1}^{\infty} 2^{n-1}\left(\frac{1}{3^{n+1}}\right) = \frac{1}{3}$$

and hence the sum of the lengths of the intervals making up each C_n now tends to $\frac{2}{3}$.

Exercise 7.6.19. As a final gesture, provide the example advertised in Exercise 7.6.13 of an integrable function f and a continuous function g where the composition $f \circ g$ is properly defined but not integrable. Exercise 4.3.12 may be useful.

Solution. Let $F \subseteq [0,1]$ be the non-zero measure Cantor-type set defined in the text (such sets are sometimes called Smith-Volterra-Cantor sets, or fat Cantor sets). Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

and note that f is integrable on any interval $[a,b]$. Define $g : [0,1] \to \mathbf{R}$ by

$$g(x) = \inf\{|x - a| : a \in F\}.$$

Exercise 4.3.12 shows that g is continuous and, because F is closed, satisfies $g(x) = 0$ if and only if $x \in F$. It follows that $f \circ g : [0,1] \to \mathbf{R}$ is given by

$$f(g(x)) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \notin F. \end{cases}$$

Using that F is closed and does not contain any intervals, we can argue as we did in Exercise 7.3.9 (d) to show that the set of discontinuities of $f \circ g$ is precisely F. As F does not have measure zero, Lebesgue's Theorem allows us to conclude that $f \circ g$ is not integrable on $[0,1]$.

Chapter 8. Additional Topics

8.1. The Generalized Riemann Integral

Exercise 8.1.1.

(a) Explain why both the Riemann sum $R(f,P)$ and $\int_a^b f$ fall between $L(f,P)$ and $U(f,P)$.

(b) Explain why $U(f,P') - L(f,P') < \varepsilon/3$.

Solution.

(a) The inequality $L(f,P) \le R(f,P) \le U(f,P)$ follows as $m_k \le f(c_k) \le M_k$ for each k and the inequality $L(f,P) \le \int_a^b f \le U(f,P)$ follows by observing that:
- $L(f,P) \le L(f)$;
- $U(f) \le U(f,P)$;
- $\int_a^b f = L(f) = U(f)$ (as f is integrable on $[a,b]$).

(b) Because P' is a refinement of P, Lemma 7.2.3 shows that $U(f,P') - L(f,P') < \varepsilon/3$.

Exercise 8.1.2. Explain why $U(f,P) - U(f,P') \ge 0$.

Solution. This follows from Lemma 7.2.3, as P' is a refinement of P.

Exercise 8.1.3.

(a) In terms of n, what is the largest number of terms of the form $M_k(x_k - x_{k-1})$ that could appear in one of $U(f,P)$ or $U(f,P')$ but not the other?

(b) Finish the proof in this direction by arguing that

$$U(f,P) - U(f,P') < \varepsilon/3.$$

Solution.

(a) Note that

$$|P'| = |P_\varepsilon| + |P| - |P_\varepsilon \cap P| = n + 1 + |P| - |P_\varepsilon \cap P|.$$

To maximize the number of points in P', the above expression shows that $|P_\varepsilon \cap P|$ should be minimized. Since both P_ε and P must contain the points a and b, the smallest this intersection could be is $|P_\varepsilon \cap P| = 2$ and thus

$$|P'| = n - 1 + |P|$$

is the largest that $|P'|$ could be. In other words, after forming P' by adding new points from P_ε to P, the largest number of points that could have been added is $n-1$. For each of these new points added, two terms are added to $U(f, P')$ which do not appear in $U(f, P)$ and there is one term in $U(f, P)$ which does not appear in $U(f, P')$. It follows that the largest number of terms that could appear in one of $U(f, P)$ or $U(f, P')$ but not the other is $3(n-1)$.

(b) Let us write $U(f, P) - U(f, P')$ as

$$\sum M_k(x_k - x_{k-1}) - \sum M_k(x_k - x_{k-1}),$$

where the sum on the left consists of those terms appearing in $U(f, P)$ but not in $U(f, P')$ and the sum on the right consists of those terms appearing in $U(f, P')$ but not in $U(f, P)$; terms which appear in both $U(f, P)$ and $U(f, P')$ cancel. By part (a), there can be at most $3(n-1)$ terms in total across both sums. Exercise 8.1.2 shows that the quantity $U(f, P) - U(f, P')$ is non-negative and thus

$$\begin{aligned} U(f, P) - U(f, P') &= |U(f, P) - U(f, P')| \\ &= \left|\sum M_k(x_k - x_{k-1}) - \sum M_k(x_k - x_{k-1})\right| \\ &\leq \sum |M_k|(x_k - x_{k-1}) + \sum |M_k|(x_k - x_{k-1}). \end{aligned}$$

Now we can use the fact that both partitions P and P' are δ-fine, that M is a bound on $|f|$, and that there are at most $3(n-1)$ terms in total across both sums to see that

$$U(f, P) - U(f, P') \leq 3(n-1)M\delta < \frac{\varepsilon}{3}.$$

Exercise 8.1.4.

(a) Show that if f is continuous, then it is possible to pick tags $\{c_k\}_{k=1}^n$ so that

$$R(f, P) = U(f, P).$$

Similarly, there are tags for which $R(f, P) = L(f, P)$ as well.

(b) If f is not continuous, it may not be possible to find tags for which

$$R(f, P) = U(f, P).$$

Show, however, that given an arbitrary $\varepsilon > 0$, it is possible to pick tags for P so that

$$U(f, P) - R(f, P) < \varepsilon.$$

The analogous statement holds for lower sums.

Solution.

(a) For $k \in \{1, \ldots, n\}$, the Extreme Value Theorem (Theorem 4.4.2) implies that f attains its maximum on the compact set $[x_{k-1}, x_k]$, i.e. there is some $c_k \in [x_{k-1}, x_k]$ such that

$f(c_k) = M_k$. Thus choosing the collection of tags $\{c_k\}_{k=1}^n$ gives us $R(f, P) = U(f, P)$. Similarly, the Extreme Value Theorem implies that f attains its minimum on $[x_{k-1}, x_k]$ at some $c_k \in [x_{k-1}, x_k]$; choosing the tags $\{c_k\}_{k=1}^n$ gives us $R(f, P) = L(f, P)$.

(b) Suppose $P = \{x_0, ..., x_n\}$. By Lemma 1.3.8, for each $k \in \{1, ..., n\}$, there exists some $c_k \in [x_{k-1}, x_k]$ such that

$$M_k - \frac{\varepsilon}{b-a} < f(c_k) \leq M_k.$$

Choose the tags $\{c_k\}_{k=1}^n$ and then observe that

$$U(f, P) - R(f, P) = \sum_{k=1}^{n} (M_k - f(c_k))\Delta x_k < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \varepsilon.$$

The analogous statement for lower sums can be proved similarly.

Exercise 8.1.5. Use the results of the previous exercise to finish the proof of Theorem 8.1.2.

Solution. See Exercise 7.2.6.

Exercise 8.1.6. Consider the interval $[0, 1]$.

(a) If $\delta(x) = 1/9$, find a $\delta(x)$-fine tagged partition of $[0, 1]$. Does the choice of tags matter in this case?

(b) Let

$$\delta(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ x/3 & \text{if } 0 < x \leq 1. \end{cases}$$

Construct a $\delta(x)$-fine tagged partition of $[0, 1]$.

Solution.

(a) Take $P = \left\{0, \frac{1}{10}, \frac{2}{10}, ..., \frac{9}{10}, 1\right\}$ and for each $k \in \{1, ..., 10\}$ choose the tag $c_k = \frac{k}{10}$. Then

$$\Delta x_k = \frac{1}{10} < \frac{1}{9} = \delta(c_k).$$

The choice of tags is irrelevant here since the gauge δ is constant.

(b) Let $P = \left\{0, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}, ..., \frac{14}{15}, 1\right\}$ and choose the tags $c_1 = 0$ and $c_k = \frac{k+2}{15}$ for $2 \leq k \leq 13$. Some tedious calculations show that this tagged partition is $\delta(x)$-fine.

Exercise 8.1.7. Finish the proof of Theorem 8.1.5.

Solution. Denote the two halves by

$$J_1 = \left[a, \frac{a+b}{2}\right] \quad \text{and} \quad J_2 = \left[\frac{a+b}{2}, b\right].$$

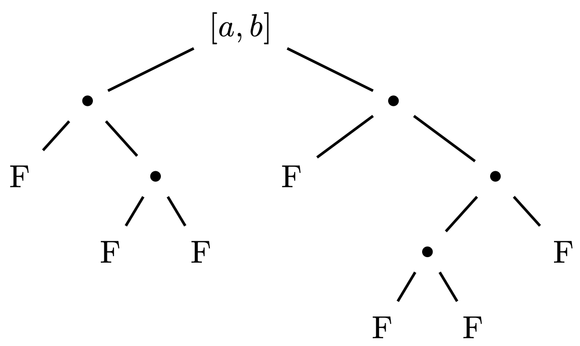
If there exists a $c_1 \in J_1$ and a $c_2 \in J_2$ such that $\frac{b-a}{2} < \delta(c_1)$ and $\frac{b-a}{2} < \delta(c_2)$, then the tagged partition

$$\left(\left\{a, \frac{a+b}{2}, b\right\}, \{c_1, c_2\}\right)$$

is $\delta(x)$-fine. Otherwise, at least one of the following statements is true:

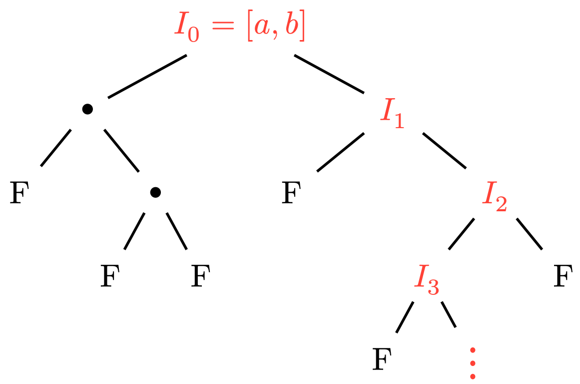
- for all $x \in J_1, \delta(x) \le \frac{b-a}{2}$;
- for all $x \in J_2, \delta(x) \le \frac{b-a}{2}$.

For each of the subintervals for which the relevant statement above is true, we perform the same procedure: bisect the interval into two equal halves and look for valid tags. By continuing this algorithm, we form a "tree" like so:



Each node of this tree (other than the topmost) represents a closed and bounded interval which is exactly half of its parent node: the left half if the node is to the left of its parent and the right half if the node is to the right of its parent; note that the length of each node is exactly half of the length of its parent node. An "F" indicates that we found a valid tag at that node, i.e. if the node is an interval J, then we found some $c \in J$ such that $|J| < \delta(c)$.

If this algorithm stops after a finite number of steps, i.e. if the tree is finite, then we may take the collection of endpoints of the terminal nodes and the tags we found there as our $\delta(x)$-fine partition. The alternative is that the algorithm does not terminate after a finite number of steps, like so:



We will show that this cannot happen. Indeed, if the algorithm fails to terminate, then we obtain a nested sequence of closed and bounded intervals (I_n) (by following the branches of the tree downwards, e.g. the red path in the figure above; there may be more than one such path) such that:

(i) $|I_n| = 2^{-n}(b-a) \to 0$;

(ii) for all $x \in I_n$ we have $\delta(x) \leq |I_n|$.

It then follows from the Nested Interval Property (Theorem 1.4.1) that there exists some $x_0 \in \bigcap_{n=1}^{\infty} I_n$. Property (ii) shows that $\delta(x_0) \leq |I_n|$ for all $n \in \mathbf{N}$ and property (i) then shows that $\delta(x_0) \leq 0$, contradicting that δ is a gauge. Hence it must be the case that the algorithm stops after a finite number of steps, yielding a $\delta(x)$-fine tagged partition.

Exercise 8.1.8. Finish the argument.

Solution. Let $\varepsilon > 0$ be given. Because f has generalized Riemann integrals A_1 and A_2, there exist gauges δ_1 and δ_2 such that

(i) for each tagged partition $\left(P, \{c_k\}_{k=1}^{n}\right)$ that is δ_1-fine, the inequality $|R(f,P) - A_1| < \frac{\varepsilon}{2}$ holds;

(ii) for each tagged partition $\left(P, \{c_k\}_{k=1}^{n}\right)$ that is δ_2-fine, the inequality $|R(f,P) - A_2| < \frac{\varepsilon}{2}$ holds;

Let $\delta : [a,b] \to \mathbf{R}$ be the gauge on $[a,b]$ given by $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. By Theorem 8.1.5, there exists a tagged partition $\left(P, \{c_k\}_{k=1}^{n}\right)$ that is δ-fine; it is straightforward to verify that this tagged partition is also δ_1- and δ_2-fine. It then follows from (i) and (ii) that

$$|A_1 - A_2| \leq |R(f,P) - A_1| + |R(f,P) - A_2| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we may conclude that $A_1 = A_2$.

Exercise 8.1.9. Explain why every function that is Riemann-integrable with $\int_a^b f = A$ must also have generalized Riemann integral A.

Solution. For any $\varepsilon > 0$ we can simply take the gauge on $[a,b]$ to be the constant function whose value is the δ supplied by Theorem 8.1.2.

Exercise 8.1.10. Show that if $\left(P, \{c_k\}_{k=1}^{n}\right)$ is a $\delta(x)$-fine tagged partition, then $R(g,P) < \varepsilon$.

Solution. Suppose $P = \{x_0, \ldots, x_n\}$. If each c_k is irrational then $R(g,P) = 0 < \varepsilon$. Otherwise, let $\{c_{k_1}, \ldots, c_{k_m}\}$ be the collection of rational tags, so that for each $1 \leq j \leq m$ we have $c_{k_j} = r_{i_j}$ for some (not necessarily unique) i_j. It follows that

$$R(g,P)=\sum_{k=1}^{n} g(c_k)\Delta x_k=\sum_{j=1}^{m}\Delta x_{k_j}<\sum_{j=1}^{m}\delta(c_{k_j})=\sum_{j=1}^{m}\delta(r_{i_j})=\sum_{j=1}^{m}\frac{\varepsilon}{2^{i_j+1}}.$$

Since a tag can appear in at most two subintervals, any given rational number r_i can appear at most twice in the collection $\{r_{i_1},...,r_{i_m}\}$. Thus

$$R(g,P)<\sum_{j=1}^{m}\frac{\varepsilon}{2^{i_j+1}}\leq 2\sum_{i=1}^{\infty}\frac{\varepsilon}{2^{i+1}}=\varepsilon.$$

Exercise 8.1.11. Show that

$$F(b)-F(a)=\sum_{k=1}^{n}[F(x_k)-F(x_{k-1})].$$

Solution. This is a telescoping sum:

$$\sum_{k=1}^{n}[F(x_k)-F(x_{k-1})]=F(x_n)-F(x_0)=F(b)-F(a).$$

Exercise 8.1.12. For each $c\in[a,b]$, explain why there exists a $\delta(c)>0$ (a $\delta>0$ depending on c) such that

$$\left|\frac{F(x)-F(c)}{x-c}-f(c)\right|<\varepsilon$$

for all $0<|x-c|<\delta(c)$.

Solution. By assumption the function F is differentiable at c and satisfies $F'(c)=f(c)$; the existence of such a $\delta(c)$ is then immediate from the definition of $F'(c)$ as

$$\lim_{x\to}\frac{F(x)-F(c)}{x-c}.$$

Exercise 8.1.13.

(a) For a particular $c_k\in[x_{k-1},x_k]$ of P, show that

$$|F(x_k)-F(c_k)-f(c_k)(x_k-c_k)|<\varepsilon(x_k-c_k)$$

and

$$|F(c_k)-F(x_{k-1})-f(c_k)(c_k-x_{k-1})|<\varepsilon(c_k-x_{k-1}).$$

(b) Now, argue that

$$|F(x_k)-F(x_{k-1})-f(c_k)(x_k-x_{k-1})|<\varepsilon(x_k-x_{k-1}),$$

and use this fact to complete the proof of the theorem.

Solution.

(a) The inequalities here should not be strict; the first strict inequality fails if $c_k = x_k$ and the second fails if $c_k = x_{k-1}$. Instead, we'll show that

$$|F(x_k) - F(c_k) - f(c_k)(x_k - c_k)| \le \varepsilon(x_k - c_k)$$
$$\text{and} \quad |F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| \le \varepsilon(c_k - x_{k-1}),$$

which is sufficient for the proof. For the first inequality, note that both sides are zero if $c_k = x_k$. Suppose therefore that $x_{k-1} \le c_k < x_k$ and notice that, because the tagged partition $(P, \{c_k\})$ is δ-fine, we have $0 < x_k - c_k \le \Delta x_k < \delta(c_k)$. It follows from Exercise 8.1.12 that

$$\left| \frac{F(x_k) - F(c_k)}{x_k - c_k} - f(c_k) \right| < \varepsilon.$$

Multiplying through by $x_k - c_k$, which is positive, gives the desired inequality. The second inequality is obtained similarly.

(b) Expressing the inequalities from part (a) as

$$-\varepsilon(x_k - c_k) \le F(x_k) - F(c_k) - f(c_k)(x_k - c_k) \le \varepsilon(x_k - c_k)$$
$$-\varepsilon(c_k - x_{k-1}) \le F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1}) \le \varepsilon(c_k - x_{k-1})$$

and adding the rows together, we see that

$$-\varepsilon(x_k - x_{k-1}) \le F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1}) \le \varepsilon(x_k - x_{k-1}).$$

Thus

$$|F(b) - F(a) - R(f, P)| \le \sum_{k=1}^{n} |F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})|$$
$$\le \sum_{k=1}^{n} \varepsilon(x_k - x_{k-1}) = \varepsilon(b - a)$$

By replacing ε in the proof with $\frac{\varepsilon}{2(b-a)}$, we obtain

$$|F(b) - F(a) - R(f, P)| < \varepsilon,$$

as desired.

Exercise 8.1.14.

(a) Why are we sure that f and $(F \circ g)'$ have generalized Riemann integrals?

(b) Use Theorem 8.1.9 to finish the proof.

Solution.

(a) Both $f = F'$ and $(F \circ g)'$ are derivatives. Theorem 8.1.9 shows that any derivative is generalized-Riemann-integrable.

(b) By the chain rule and Theorem 8.1.9:

$$\int_a^b (f \circ g) \cdot g' = \int_a^b (F' \circ g) \cdot g' = \int_a^b (F \circ g)' = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} F' = \int_{g(a)}^{g(b)} f.$$

8.2. Metric Spaces and the Baire Category Theorem

Exercise 8.2.1. Decide which of the following are metrics on $X = \mathbf{R}^2$. For each, we let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in the plane.

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(b) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(c) $d(x, y) = |x_1 x_2 + y_1 y_2|$.

Solution.

(a) This is a metric on $\mathbf{R}^2$. To see this, we shall verify each property in Definition 8.2.1. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbf{R}^2$ be given.

(i) It is clear that $d(x, y) \geq 0$. Observe that

$$\begin{aligned} d(x,y) = 0 \quad &\Leftrightarrow \quad \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0 \\ &\Leftrightarrow \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 = 0 \\ &\Leftrightarrow \quad (x_1 - y_1)^2 = 0 \text{ and } (x_2 - y_2)^2 = 0 \\ &\Leftrightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2 \\ &\Leftrightarrow \quad x = y. \end{aligned}$$

(ii) We have

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = d(y, x).$$

(iii) For $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbf{R}^2$, observe that

$$\begin{aligned} &\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \\ \Leftrightarrow \quad &(a_1 + b_1)^2 + (a_2 + b_2)^2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \\ \Leftrightarrow \quad &a_1 b_1 + a_2 b_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}. \end{aligned}$$

This last inequality follows from the Cauchy-Schwarz inequality. The desired triangle inequality for d can now be obtained by taking $a = x - z$ and $b = z - y$.

(b) This is a metric on $\mathbf{R}^2$. To see this, we shall verify each property in Definition 8.2.1. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbf{R}^2$ be given.

(i) It is clear that $d(x, y) \geq 0$. Observe that

$$\begin{aligned} d(x,y)=0 &\Leftrightarrow \max\{|x_1-y_1|,|x_2-y_2|\}=0 \\ &\Leftrightarrow |x_1-y_1|=0 \text{ and } |x_2-y_2|=0 \\ &\Leftrightarrow x_1=y_1 \text{ and } x_2=y_2 \\ &\Leftrightarrow x=y. \end{aligned}$$

(ii) We have

$$d(x,y)=\max\{|x_1-y_1|,|x_2-y_2|\}=\max\{|y_1-x_1|,|y_2-x_2|\}=d(y,x).$$

(iii) Let $z=(z_1,z_2)\in\mathbf{R}^2$ be given. Suppose that $d(x,y)=|x_1-y_1|$ (the case where $d(x,y)=|x_2-y_2|$ is handled similarly) and observe that

$$d(x,y)=|x_1-y_1|\le|x_1-z_1|+|z_1-y_1|\le d(x,z)+d(z,y).$$

(c) This is not a metric on $\mathbf{R}^2$. To see this, observe that by taking $x=(1,1)$ and $y=(-1,1)$ we obtain $d(x,y)=0$, but $x\neq y$. Thus property (i) of Definition 8.2.1 is not satisfied.

Exercise 8.2.2. Let $C[0,1]$ be the collection of continuous functions on the closed interval $[0,1]$. Decide which of the following are metrics on $C[0,1]$.

(a) $d(f,g)=\sup\{|f(x)-g(x)|:x\in[0,1]\}$.

(b) $d(f,g)=|f(1)-g(1)|$.

(c) $d(f,g)=\int_0^1|f-g|$.

Solution.

(a) This is a metric on $C[0,1]$. Note that by the Extreme Value Theorem (Theorem 4.4.2), the supremum is actually a maximum.

(i) Because each element of $\{|f(x)-g(x)|:x\in[0,1]\}$ is non-negative, we must have $d(f,g)\ge0$. Observe that

$$\begin{aligned} d(f,g)=0 &\Leftrightarrow \max\{|f(x)-g(x)|:x\in[0,1]\}=0 \\ &\Leftrightarrow |f(x)-g(x)|=0 \text{ for all } x\in[0,1] \\ &\Leftrightarrow f(x)=g(x) \text{ for all } x\in[0,1] \\ &\Leftrightarrow f=g. \end{aligned}$$

(ii) As $|f(x)-g(x)|=|g(x)-f(x)|$ for each $x\in[0,1]$, we see that $d(f,g)=d(g,f)$.

(iii) Let $h\in C[0,1]$ be given and suppose that $|f-g|$ attains its maximum at some $t\in[0,1]$, so that $d(f,g)=|f(t)-g(t)|$. Then:

$$d(f,g)=|f(t)-g(t)|\le|f(t)-h(t)|+|h(t)-g(t)|\le d(f,h)+d(h,g).$$

(b) This is not a metric on $C[0,1]$. To see this, let $f,g\in C[0,1]$ be given by $f(x)=0$ and $g(x)=1-x$. Then

$$d(f,g) = |f(1) - g(1)| = 0$$

but $f \neq g$. Thus d does not satisfy property (i) in Definition 8.2.1.

(c) This is a metric on $C[0,1]$:

(i) As $|f-g| \geq 0$, Theorem 7.4.2 (iv) shows that $d(f,g) \geq 0$. Observe that

$$\begin{aligned} d(f,g) = 0 \quad &\Leftrightarrow \quad \int_0^1 |f-g| = 0 \\ &\Leftrightarrow \quad |f(x) - g(x)| = 0 \text{ for all } x \in [0,1] \\ &\Leftrightarrow \quad f(x) = g(x) \text{ for all } x \in [0,1] \\ &\Leftrightarrow \quad f = g, \end{aligned}$$

where we have used the contrapositive of Exercise 7.4.3 (c) for the second equivalence.

(ii) We have $d(f,g) = d(g,f)$ since $|f-g| = |g-f|$.

(iii) Let $h \in C[0,1]$ be given. For any $x \in [0,1]$ we have the inequality

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|.$$

Theorem 7.4.2 (iv) then implies that

$$\int_0^1 |f-g| \leq \int_0^1 |f-h| + \int_0^1 |h-g|,$$

i.e. $d(f,g) \leq d(f,h) + d(h,g)$.

Exercise 8.2.3. Verify that the discrete metric is actually a metric.

Solution. Properties (i) and (ii) in Definition 8.2.1 are clear. For the triangle inequality, let $x, y, z \in X$ be given and suppose that all three are distinct. Then

$$\rho(x,y) = 1 < 2 = \rho(x,z) + \rho(z,y).$$

Now suppose that $x \neq y$ and $y = z$. Then

$$\rho(x,y) = 1 = \rho(x,z) + \rho(z,y).$$

The other cases are handled similarly.

Exercise 8.2.4. Show that a convergent sequence is Cauchy.

Solution. Suppose that (x_n) is sequence in a metric space (X,d) converging to some $x \in X$ and let $\varepsilon > 0$ be given. There exists an $N \in \mathbf{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ whenever $n \geq N$. Suppose that $m, n \geq N$ and observe that, by the triangle inequality for d,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon.$$

Thus (x_n) is Cauchy.

Exercise 8.2.5.

(a) Consider $\mathbf{R}^2$ with the discrete metric $\rho(x, y)$ examined in Exercise 8.2.3. What do Cauchy sequences look like in this space? Is $\mathbf{R}^2$ complete with respect to this metric?

(b) Show that $C[0, 1]$ is complete with respect to the metric in Exercise 8.2.2 (a).

(c) Define $C^1[0, 1]$ to be the collection of differentiable functions on $[0, 1]$ whose derivatives are also continuous. Is $C^1[0, 1]$ complete with respect to the metric defined in Exercise 8.2.2 (a)?

Solution.

(a) Suppose (x_n) is a Cauchy sequence in $(\mathbf{R}^2, \rho)$. There exists an $N \in \mathbf{N}$ such that $\rho(x_m, x_n) < \frac{1}{2}$ for any $m, n \geq N$. Since $\rho(x, y) \in \{0, 1\}$, we have $\rho(x, y) < \frac{1}{2}$ if and only if $\rho(x, y) = 0$, which is the case if and only if $x = y$. Thus $x_m = x_n$ for all $m, n \geq N$. In particular, $x_n = x_N$ for all $n \geq N$, i.e. the sequence (x_n) is eventually constant. It is straightforward to prove that eventually constant sequences converge to that constant (in any metric space) and thus $(\mathbf{R}^2, \rho)$ is a complete metric space.

(b) Let d be the metric from Exercise 8.2.2 (a). Here is a useful lemma, the proof of which is essentially immediate from the definitions.

Lemma L.18. Suppose (f_n) is a sequence of functions in $C[a, b]$ and $f \in C[a, b]$. Then (f_n) converges to f in the metric space $(C[a, b], d)$ (in the sense of Definition 8.2.2) if and only if (f_n) converges to f uniformly (in the sense of Definition 6.2.3).

Now suppose that (f_n) is a Cauchy sequence in $(C[0, 1], d)$ and let $\varepsilon > 0$ be given. There exists an $N \in \mathbf{N}$ such that $d(f_m, f_n) < \varepsilon$ whenever $m, n \geq N$. Thus, for any $m, n \geq N$ and any $x \in [0, 1]$, we have

$$|f_m(x) - f_n(x)| \leq d(f_m, f_n) < \varepsilon.$$

It follows from Theorem 6.2.5 that there is a function $f : [0, 1] \to \mathbf{R}$ such that $f_n \to f$ uniformly; note that f must belong to $C[0, 1]$ by Theorem 6.2.6. Lemma L.18 now implies that (f_n) converges to f in the metric space $(C[0, 1], d)$ and we may conclude that this metric space is complete.

(c) This metric space is not complete. To see this, consider the sequence of functions (f_n) in $C^1[0, 1]$ given by $f_n(x) = \sqrt{x + \frac{1}{n}}$. We claim that this is a Cauchy sequence in $(C^1[0, 1], d)$. For a given $\varepsilon > 0$ let $N \in \mathbf{N}$ be such that $N > 4\varepsilon^{-2}$ and suppose that $n \geq m \geq N$. Then for any $x \in [0, 1]$ we have

$$|f_m(x) - f_n(x)| = \sqrt{x + \frac{1}{m}} - \sqrt{x + \frac{1}{n}} = \frac{\frac{1}{m} - \frac{1}{n}}{\sqrt{x + \frac{1}{m}} + \sqrt{x + \frac{1}{n}}}$$
$$\leq \frac{\frac{1}{m}}{\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}} = \frac{\frac{1}{\sqrt{m}}}{1 + \frac{\sqrt{m}}{\sqrt{n}}} \leq \frac{1}{\sqrt{m}} < \frac{\varepsilon}{2}.$$

As $x \in [0,1]$ was arbitrary, we see that

$$n \geq m \geq N \quad \Rightarrow \quad d(f_m, f_n) \leq \frac{\varepsilon}{2} < \varepsilon$$

and our claim follows.

Next we claim that (f_n) is not a convergent sequence in $(C^1[0,1], d)$. To see this, we will argue by contradiction: suppose that there is some $f \in C^1[0,1]$ such that $d(f_n, f) \to 0$. Fix $x \in [0,1]$ and observe that $|f_n(x) - f(x)| \leq d(f_n, f)$; the Squeeze Theorem then implies that the sequence of real numbers $(f_n(x))$ converges to $f(x)$ (i.e. in the metric space $\mathbf{R}$ with the usual metric). On the other hand, it is evident that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x + \frac{1}{n}} = \sqrt{x}.$$

Since limits are unique (Theorem 2.2.7; this actually holds in any metric space), we must have $f(x) = \sqrt{x}$ for each $x \in [0,1]$—but this implies that f is not differentiable at $x = 0$, contradicting that $f \in C^1[0,1]$. We must conclude that (f_n) does not converge in the metric space $(C^1[0,1], d)$.

Exercise 8.2.6. Which of these functions from $C[0,1]$ to $\mathbf{R}$ (with the usual metric) are continuous?

(a) $g(f) = \int_0^1 fk$, where k is some fixed function in $C[0,1]$.

(b) $g(f) = f(1/2)$.

(c) $g(f) = f(1/2)$, but this time with respect to the metric on $C[0,1]$ from Exercise 8.2.2 (c).

Solution.

(a) This function is continuous. Fix $f \in C[0,1]$, let $\varepsilon > 0$ be given, and let

$$\delta = \frac{\varepsilon}{1 + \int_0^1 |k|}.$$

Then for any $h \in C[0,1]$ satisfying $d(f,h) < \delta$ we have

$$|g(f) - g(h)| = \left|\int_0^1 fk - \int_0^1 hk\right| = \left|\int_0^1 (f-h)k\right| \leq d(f,h) \int_0^1 |k| < \delta \int_0^1 |k| < \varepsilon.$$

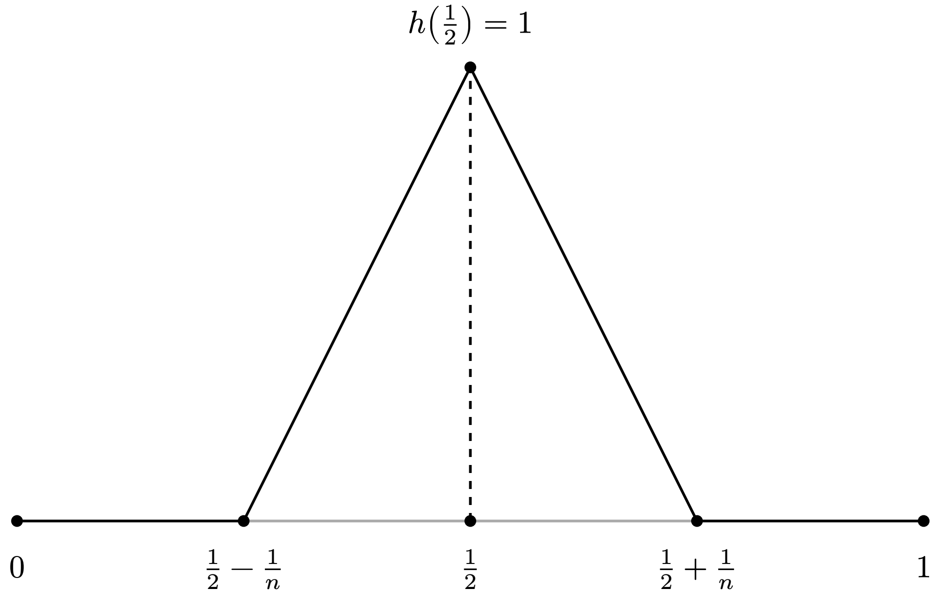
Thus g is continuous at any $f \in C[0,1]$.

(b) This function is continuous. Fix $f \in C[0,1]$, let $\varepsilon > 0$ be given, and let $\delta = \varepsilon$. Then for any $h \in C[0,1]$ satisfying $d(f,h) < \delta$ we have

$$|g(f) - g(h)| = \left| f\left(\frac{1}{2}\right) - h\left(\frac{1}{2}\right) \right| \leq d(f,h) < \varepsilon.$$

Thus g is continuous at any $f \in C[0,1]$.

(c) This function is not continuous: we will show that g is not continuous at the constant function $f(x) = 0$. For any $\delta > 0$ let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \delta$ and $n \geq 3$. Define $h : [0,1] \to \mathbf{R}$ to be the continuous piecewise-linear function passing through the points $(0,0), \left(\frac{1}{2} - \frac{1}{n}, 0\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2} + \frac{1}{n}, 0\right)$, and $(1,0)$; see the figure below.



Then

$$d(f,h) = \int_0^1 |f - h| = \int_0^1 h = \frac{1}{n} < \delta$$

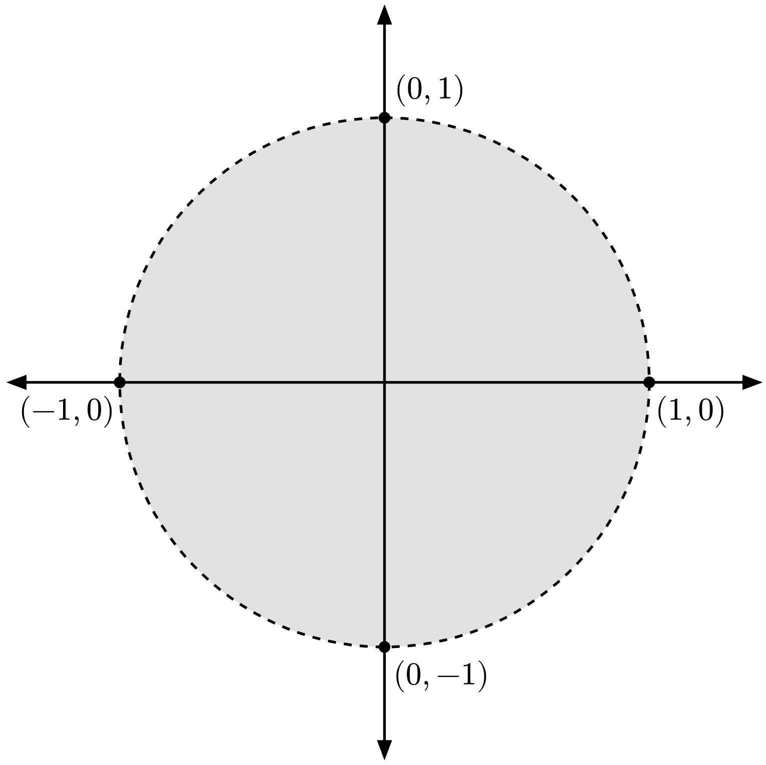
but $|g(f) - g(h)| = \left|f\left(\frac{1}{2}\right) - h\left(\frac{1}{2}\right)\right| = 1$. Thus g is not continuous at f.

Exercise 8.2.7. Describe the ε-neighborhoods in $\mathbf{R}^2$ for each of the different metrics described in Exercise 8.2.1. How about for the discrete metric?

Solution. Let d be the metric from Exercise 8.2.1 (a) and let d' be the metric from Exercise 8.2.1 (b). With respect to d, a typical ε-neighborhood of some $x = (x_1, x_2) \in \mathbf{R}^2$ is the set

$$V_\varepsilon(x) = \left\{ y = (y_1, y_2) \in \mathbf{R}^2 : \sqrt{(x_1 - y_1)^2 - (x_2 - y_2)^2} < \varepsilon \right\}.$$

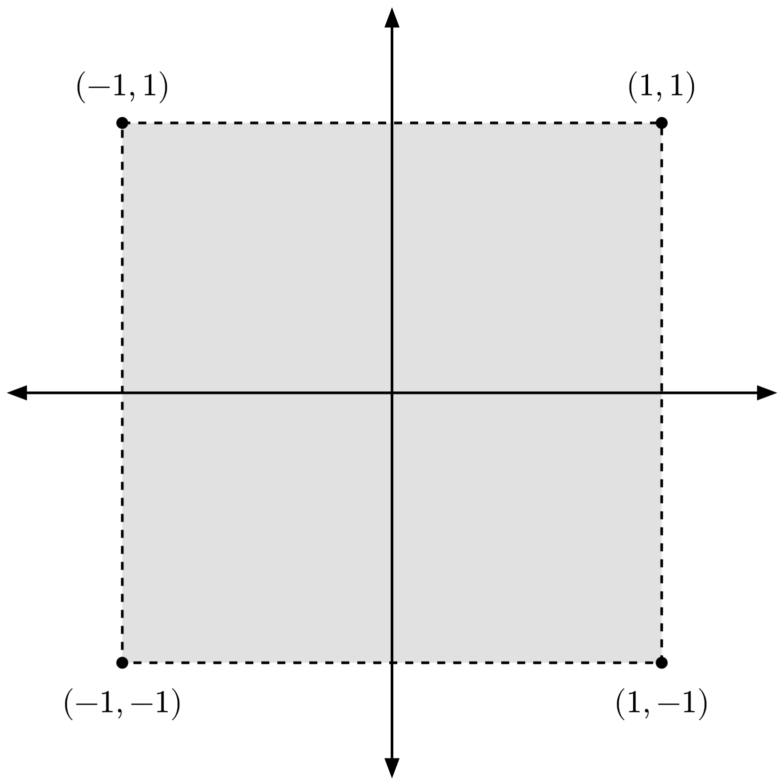
This consists of all the points contained strictly inside the circle of radius ε centred at x; see the following figure, which shows $V_1(0)$ with respect to d.



With respect to d', a typical ε-neighborhood of some $x = (x_1, x_2) \in \mathbf{R}^2$ is the set

$$V_\varepsilon(x) = \{y = (y_1, y_2) \in \mathbf{R}^2 : \max\{|x_1 - y_1|, |x_2 - y_2|\} < \varepsilon\}.$$

This consists of all the points contained strictly inside the square of side length 2ε centred at x; see the figure below, which shows $V_1(0)$ with respect to d'.



For the discrete metric ρ, we have

$$V_\varepsilon(x) = \begin{cases} \{x\} & \text{if } 0 < \varepsilon \le 1, \\ \mathbf{R}^2 & \text{if } \varepsilon > 1. \end{cases}$$

This situation is typical for a discrete metric space.

Exercise 8.2.8. Let (X, d) be a metric space.

(a) Verify that a typical ε-neighborhood $V_\varepsilon(x)$ is an open set. Is the set

$$C_\varepsilon(x) = \{y \in X : d(x, y) \le \varepsilon\}$$

a closed set?

(b) Show that a set $E \subseteq X$ is open if and only if its complement is closed.

Solution.

(a) Let $\varepsilon > 0$ and $x \in X$ be given. For $y \in V_\varepsilon(x)$, let $\delta = \varepsilon - d(x, y) > 0$. We claim that $V_\delta(y) \subseteq V_\varepsilon(x)$. Indeed, suppose that $z \in V_\delta(y)$, so that

$$d(z, y) < \delta = \varepsilon - d(x, y) \quad \Leftrightarrow \quad d(z, y) + d(x, y) < \varepsilon.$$

The triangle inequality then gives us

$$d(z, x) \le d(z, y) + d(x, y) < \varepsilon.$$

Thus $z \in V_\varepsilon(x)$ and it follows that $V_\delta(y) \subseteq V_\varepsilon(x)$. Hence $V_\varepsilon(x)$ is an open set.

Now we will show that, for $\varepsilon > 0$ and $x \in X$, the set $C_\varepsilon(x)$ is closed. To see this, let us prove the following:

if $y \in X$ is such that $d(x, y) > \varepsilon$ then y is not a limit point of $C_\varepsilon(x)$.

Let $\delta = d(x, y) - \varepsilon > 0$ and suppose $z \in V_\delta(y)$, so that

$$d(z, y) < \delta = d(x, y) - \varepsilon \quad \Leftrightarrow \quad d(x, y) - d(z, y) > \varepsilon.$$

It follows from the triangle inequality that

$$d(x, y) \le d(z, x) + d(z, y) \quad \Rightarrow \quad d(z, x) \ge d(x, y) - d(z, y) > \varepsilon.$$

Thus $d(z, x) > \varepsilon$, so that $z \notin C_\varepsilon(x)$. We have now shown that there is a $\delta > 0$ such that $V_\delta(y) \cap C_\varepsilon(x) = \emptyset$. It follows that y is not a limit point of $C_\varepsilon(x)$, as desired. The contrapositive of this statement is

if $y \in X$ is a limit point of $C_\varepsilon(x)$ then $d(x, y) \le \varepsilon$.

In other words, if y is a limit point of $C_\varepsilon(x)$ then y belongs to $C_\varepsilon(x)$. We may conclude that $C_\varepsilon(x)$ is a closed set.

(b) Observe that

$$\begin{aligned}
E \text{ is not open} \quad &\Leftrightarrow \quad (\exists x \in E)(\forall \varepsilon > 0)(V_\varepsilon(x) \not\subseteq E) \\
&\Leftrightarrow \quad (\exists x \in E)(\forall \varepsilon > 0)(V_\varepsilon(x) \cap E^c \neq \emptyset) \\
&\Leftrightarrow \quad (\exists x \in E)(\forall \varepsilon > 0)(V_\varepsilon(x) \cap (E^c \setminus \{x\}) \neq \emptyset) \\
&\Leftrightarrow \quad (\exists x \in E)(x \text{ is a limit point of } E^c) \\
&\Leftrightarrow \quad E^c \text{ does not contain all of its limit points} \\
&\Leftrightarrow \quad E^c \text{ is not closed.}
\end{aligned}$$

Exercise 8.2.9.

(a) Show that the set $Y = \left\{f \in C[0,1] : \|f\|_\infty \leq 1\right\}$ is closed in $C[0,1]$.

(b) Is the set $T = \{f \in C[0,1] : f(0) = 0\}$ open, closed, or neither in $C[0,1]$?

Solution.

(a) Using the notation of Exercise 8.2.8 (a), observe that $Y = C_1(0)$ (by 0 we mean the function that is identically zero on $[0,1]$). Thus, by Exercise 8.2.8 (a), Y is closed.

(b) T is not open. To see this, first observe that $0 \in T$. Now let $\varepsilon > 0$ be given and define $f_\varepsilon \in C[0,1]$ by $f_\varepsilon(x) = \frac{\varepsilon}{2}$. Then

$$d(f_\varepsilon, 0) = \frac{\varepsilon}{2} < \varepsilon,$$

so that $f_\varepsilon \in V_\varepsilon(0)$. However, $f_\varepsilon \notin T$ and so $V_\varepsilon(0) \not\subseteq T$. Since $\varepsilon > 0$ was arbitrary, we see that T is not open.

T is closed. To see this, suppose that $g \in C[0,1]$ is a limit point of T and let $\varepsilon > 0$ be given. There exists some $f \in V_\varepsilon(g) \cap T$ such that $f \neq g$ and it follows that

$$|g(0)| = |g(0) - f(0)| \leq d(g, f) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we see that $g(0) = 0$, so that $g \in T$. Thus T contains each of its limit points, i.e. T is closed.

Exercise 8.2.10.

(a) Supply a definition for *bounded* subsets of a metric space (X, d).

(b) Show that if K is a compact subset of the metric space (X, d), then K is closed and bounded.

(c) Show that $Y \subseteq C[0,1]$ from Exercise 8.2.9 (a) is closed and bounded but not compact.

Solution.

(a) A subset $E \subseteq X$ is bounded if there exists some $y \in X$ and some $M > 0$ such that $d(x, y) \le M$ for all $x \in E$, i.e. $E \subseteq C_M(y)$.

(b) We will prove the contrapositive statement. First, suppose that K is not closed. Then there exists some $y \notin K$ such that y is a limit point of K. Thus, for each $n \in \mathbf{N}$, there exists some $x_n \in V_{n^{-1}}(y) \cap K$, i.e. there is some $x_n \in K$ such that $d(x_n, y) < \frac{1}{n}$. Given this, it is clear that (x_n) converges to y and hence any subsequence of (x_n) converges to y (the proof of this is essentially the same as the proof of Theorem 2.5.2). Since y does not belong to K, it follows that K is not compact.

Next, suppose that K is not bounded and pick some $x_1 \in K$. Because K is not bounded, it must be the case that K is not contained in $C_1(x_1)$ and thus there exists some $x_2 \in K$ satisfying $d(x_1, x_2) > 1$. Similarly, it must be the case that K is not contained in

$$C_1(x_1) \cup C_1(x_2).$$

Thus there exists some $x_3 \in K$ satisfying $d(x_1, x_3) > 1$ and $d(x_2, x_3) > 1$. If we continue in this manner we obtain a sequence (x_n) in K such that $d(x_m, x_n) > 1$ for all $n > m$. Suppose that (x_{n_k}) is a subsequence of (x_n) and observe that for any $K \in \mathbf{N}$ we have $d\left(x_{n_K}, x_{n_{K+1}}\right) > 1$. It follows that (x_{n_k}) is not Cauchy and hence not convergent (by Exercise 8.2.4). As (x_{n_k}) was an arbitrary subsequence, we see that K is not compact.

(c) We showed in Exercise 8.2.9 (a) that Y is closed, and it is clearly bounded. To see that Y is not compact, consider the sequence of functions (f_n) given by $f_n(x) = x^n$, each of which is continuous on $[0, 1]$, satisfies $\|f_n\|_\infty = 1$, and hence belongs to Y. We will argue by contradiction to show that (f_n) has no convergent subsequence. If (f_{n_k}) is a subsequence converging to some $f \in C[0, 1]$, then in particular f is the pointwise limit of (f_{n_k}) on $[0, 1]$. However, we can see directly that the pointwise limit of (f_{n_k}) is the function

$$x \mapsto \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Since limits are unique, it must be the case that f is given by the function above, which is not continuous at $x = 1$. This contradicts $f \in C[0, 1]$.

Exercise 8.2.11.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^{\circ} = E$.

(b) Show that $\overline{E}^{c} = (E^{c})^{\circ}$, and similarly that $(E^{\circ})^{c} = \overline{E^{c}}$.

Solution.

(a) See Exercise 3.2.14 (a).

(b) See Exercise 3.2.14 (b).

Exercise 8.2.12.

(a) Show

$$\overline{V_\varepsilon(x)} \subseteq \{y \in X : d(x,y) \leq \varepsilon\},$$

in an arbitrary metric space (X, d).

(b) To keep things from sounding too familiar, find an example of a specific metric space where

$$\overline{V_\varepsilon(x)} \neq \{y \in X : d(x,y) \leq \varepsilon\}.$$

Solution.

(a) Using the notation from Exercise 8.2.8, note that $\{y \in X : d(x,y) \leq \varepsilon\} = C_\varepsilon(x)$. Clearly $V_\varepsilon(x) \subseteq C_\varepsilon(x)$ and thus if y is a limit point of $V_\varepsilon(x)$ then y is also a limit point of $C_\varepsilon(x)$. As we showed in Exercise 8.2.8, $C_\varepsilon(x)$ is closed and hence $y \in C_\varepsilon(x)$. We may conclude that $\overline{V_\varepsilon(x)} \subseteq C_\varepsilon(x)$.

(b) Consider the metric space $(\mathbf{R}, \rho)$, where ρ is the discrete metric. Then

$$\overline{V_1(0)} = \overline{\{0\}} = \overline{C_{1/2}(0)} = C_{1/2}(0) = \{0\} \neq \mathbf{R} = C_1(0).$$

Exercise 8.2.13. If E is a subset of a metric space (X, d), show that E is nowhere-dense in X if and only if $\overline{E}^c$ is dense in X.

Solution. For the purposes of this exericse, let us denote by κE the closure of E, by ιE the interior of E, and by cE the complement of E. Observe that

$$\begin{aligned} c\kappa E \text{ is dense in } X &\Leftrightarrow \kappa c \kappa E = X \\ &\Leftrightarrow c\kappa c\kappa E = \emptyset \\ &\Leftrightarrow \iota c c \kappa E = \emptyset \\ &\Leftrightarrow \iota\kappa E = \emptyset \\ &\Leftrightarrow E \text{ is nowhere-dense} \in X, \end{aligned}$$

where we have used Exercise 8.2.11 (b) for the third equivalence.

Exercise 8.2.14.

(a) Give the details for why we know there exists a point $x_2 \in V_{\varepsilon_1}(x_1) \cap O_2$ and an $\varepsilon_2 > 0$ satisfying $\varepsilon_2 < \varepsilon_1/2$ with $V_{\varepsilon_2}(x_2)$ contained in O_2 and

$$\overline{V_{\varepsilon_2}(x_2)} \subseteq V_{\varepsilon_1}(x_1).$$

(b) Proceed along this line and use the completeness of (X, d) to product a single point $x \in O_n$ for every $n \in \mathbf{N}$.

Solution.

(a) Note that x_1 must be a limit point of O_2 as O_2 is dense in X and thus there exists some $x_2 \in V_{\varepsilon_1}(x_1) \cap O_2$. Since O_2 is open there exists some $\delta > 0$ such that $V_\delta(x_2) \subseteq O_2$. If we let

$$\varepsilon_2 = \min\left\{\delta, \frac{\varepsilon_1}{4}, r := \frac{\varepsilon_1 - d(x_1, x_2)}{2}\right\},$$

then:

- $V_{\varepsilon_2}(x_2) \subseteq V_\delta(x_2) \subseteq O_2$;
- $\varepsilon_2 < \varepsilon_1/2$;
- $\overline{V_{\varepsilon_2}(x_2)} \subseteq \overline{V_r(x_2)} \subseteq C_r(x_2) \subseteq V_{\varepsilon_1}(x_1)$, where we have used Exercise 8.2.12 (a) for the second inclusion.

(b) By continuing this process, we obtain a sequence (x_n) of points in X and a sequence (ε_n) of positive real numbers such that:

(i) $\varepsilon_n < \varepsilon_1/2^{n-1}$ for each $n \geq 2$;

(ii) $V_{\varepsilon_n}(x_n) \subseteq O_n$ for each $n \in \mathbf{N}$;

(iii) the following chain of inclusions holds:

$$\cdots \subseteq V_{\varepsilon_n}(x_n) \subseteq \overline{V_{\varepsilon_n}(x_n)} \subseteq V_{\varepsilon_{n-1}}(x_{n-1}) \subseteq \overline{V_{\varepsilon_{n-1}}(x_{n-1})} \subseteq \cdots \subseteq V_{\varepsilon_2}(x_2) \subseteq \overline{V_{\varepsilon_2}(x_2)} \subseteq V_{\varepsilon_1}(x_1) \subseteq \overline{V_{\varepsilon_1}(x_1)}.$$

By (i), for any $\varepsilon > 0$ we can choose an $N \geq 2$ such that $2\varepsilon_N < \varepsilon$. Suppose $n \geq m \geq N$. By (iii) we have $x_m, x_n \in V_{\varepsilon_N}(x_N)$ and thus

$$d(x_m, x_n) \leq d(x_m, x_N) + d(x_n, x_N) < 2\varepsilon_N < \varepsilon.$$

It follows that (x_n) is a Cauchy sequence. By assumption the metric space (X, d) is complete and thus there exists some x_0 such that $\lim x_n = x_0$.

For any $m \in \mathbf{N}$, (iii) implies that the sequence (x_n) is eventually contained inside the set $\overline{V_{\varepsilon_{m+1}}(x_{m+1})}$; it follows that x_0 is a limit point of $\overline{V_{\varepsilon_{m+1}}(x_{m+1})}$. Since this set is closed, we have by (ii) and (iii):

$$x_0 \in \overline{V_{\varepsilon_{m+1}}(x_{m+1})} \subseteq V_{\varepsilon_m}(x_m) \subseteq O_m.$$

Thus $x_0 \in \bigcap_{m=1}^{\infty} O_m$.

Exercise 8.2.15. Complete the proof of the theorem.

Solution. Let (X, d) be a complete metric space and suppose $\{E_n : n \in \mathbf{N}\}$ is a countable collection of nowhere-dense sets. Notice that each $\overline{E_n}^{\mathrm{c}}$ is open (Exercise 8.2.8 (b)) and dense (Exercise 8.2.13); it follows from Theorem 8.2.10 that $\bigcap_{n=1}^{\infty} \overline{E_n}^{\mathrm{c}} \neq \emptyset$. Now observe that

$$E_n \subseteq \overline{E_n} \text{ for each } n \in \mathbf{N} \quad \Rightarrow \quad \overline{E_n}^{\mathrm{c}} \subseteq E_n^{\mathrm{c}} \text{ for each } n \in \mathbf{N} \quad \Rightarrow \quad \bigcap_{n=1}^{\infty} \overline{E_n}^{\mathrm{c}} \subseteq \bigcap_{n=1}^{\infty} E_n^{\mathrm{c}}.$$

Thus $\bigcap_{n=1}^{\infty} E_n^{\mathrm{c}} \neq \emptyset$, which implies that

$$X \neq \left(\bigcap_{n=1}^{\infty} E_n^{\mathrm{c}} \right)^{\mathrm{c}} = \bigcup_{n=1}^{\infty} E_n.$$

Exercise 8.2.16. Show that if $f \in C[0,1]$ is differentiable at a point $x \in [0,1]$, then $f \in A_{m,n}$ for some pair $m, n \in \mathbf{N}$.

Solution. By assumption we have

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

and thus there exists a $\delta > 0$ such that

$$0 < |x - t| < \delta \quad \Rightarrow \quad \left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| < 1.$$

Let $m \in \mathbf{N}$ be such that $\frac{1}{m} < \delta$ and let $n \in \mathbf{N}$ be such that $1 + |f'(x)| \leq n$. Then:

$$\begin{aligned} 0 < |x - t| < \frac{1}{m} < \delta \quad \Rightarrow \quad \left| \frac{f(x) - f(t)}{x - t} \right| &\leq \left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| + |f'(x)| \\ &< 1 + |f'(x)| \leq n. \end{aligned}$$

Thus $f \in A_{m,n}$.

Exercise 8.2.17.

(a) The sequence (x_k) does not necessarily converge, but explain why there exists a subsequence (x_{k_l}) that is convergent. Let $x = \lim(x_{k_l})$.

(b) Prove that $f_{k_l}(x_{k_l}) \to f(x)$.

(c) Now finish the proof that $A_{m,n}$ is closed.

Solution.

(a) The sequence (x_n) is contained in the interval $[0,1]$ and thus by the Bolzano-Weierstrass Theorem (Theorem 2.5.5) there exists a convergent subsequence (x_{k_l}).

(b) Let $\varepsilon > 0$ be given. As $f_k \to f$ in $C[0,1]$, there is an $L_1 \in \mathbf{N}$ such that

$$l \geq L_1 \quad \Rightarrow \quad d(f_{k_l}, f) < \tfrac{\varepsilon}{2}.$$

The continuity of f at x implies that $\lim_{l\to\infty} f(x_{k_l}) = f(x)$ and thus there is an $L_2 \in \mathbf{N}$ such that

$$l \geq L_2 \quad \Rightarrow \quad |f(x_{k_l}) - f(x)| < \tfrac{\varepsilon}{2}.$$

Now observe that for $l \geq \max\{L_1, L_2\}$ we have

$$|f_{k_l}(x_{k_l}) - f(x)| \leq |f_{k_l}(x_{k_l}) - f(x_{k_l})| + |f(x_{k_l}) - f(x)| \leq d(f_{k_l}, f) + \tfrac{\varepsilon}{2} < \varepsilon.$$

It follows that $f_{k_l}(x_{k_l}) \to f(x)$.

(c) Suppose t is such that $0 < |x - t| < \frac{1}{m}$. Because $x_{k_l} \to x$, there is an $L \in \mathbf{N}$ such that

$$l \geq L \quad \Rightarrow \quad |x - x_{k_l}| < \frac{1}{m} - |x - t| \quad \Rightarrow \quad |x_{k_l} - t| \leq |x - x_{k_l}| + |x - t| < \frac{1}{m}.$$

This implies that

$$\left| \frac{f_{k_l}(x_{k_l}) - f_{k_l}(t)}{x_{k_l} - t} \right| \leq n \quad \text{for all } l \geq L.$$

Taking the limit as $l \to \infty$ on both sides of this inequality and using part (b), we see that

$$\left| \frac{f(x) - f(t)}{x - t} \right| \leq n$$

and hence $f \in A_{m,n}$. We may conclude that $A_{m,n}$ contains its limit points and hence is closed.

Exercise 8.2.18. A continuous function is called *polygonal* if its graph consists of a finite number of line segments.

(a) Show that there exists a polygonal function $p \in C[0,1]$ satisfying $\|f - p\|_\infty < \varepsilon/2$.

(b) Show that if h is any function in $C[0,1]$ that is bounded by 1, then the function

$$g(x) = p(x) + \frac{\varepsilon}{2} h(x)$$

satisfies $g \in V_\varepsilon(f)$.

(c) Construct a polygonal function $h(x)$ in $C[0,1]$ that is bounded by 1 and leads to the conclusion $g \notin A_{m,n}$, where g is defined as in (b). Explain how this completes the argument for Theorem 8.2.12.

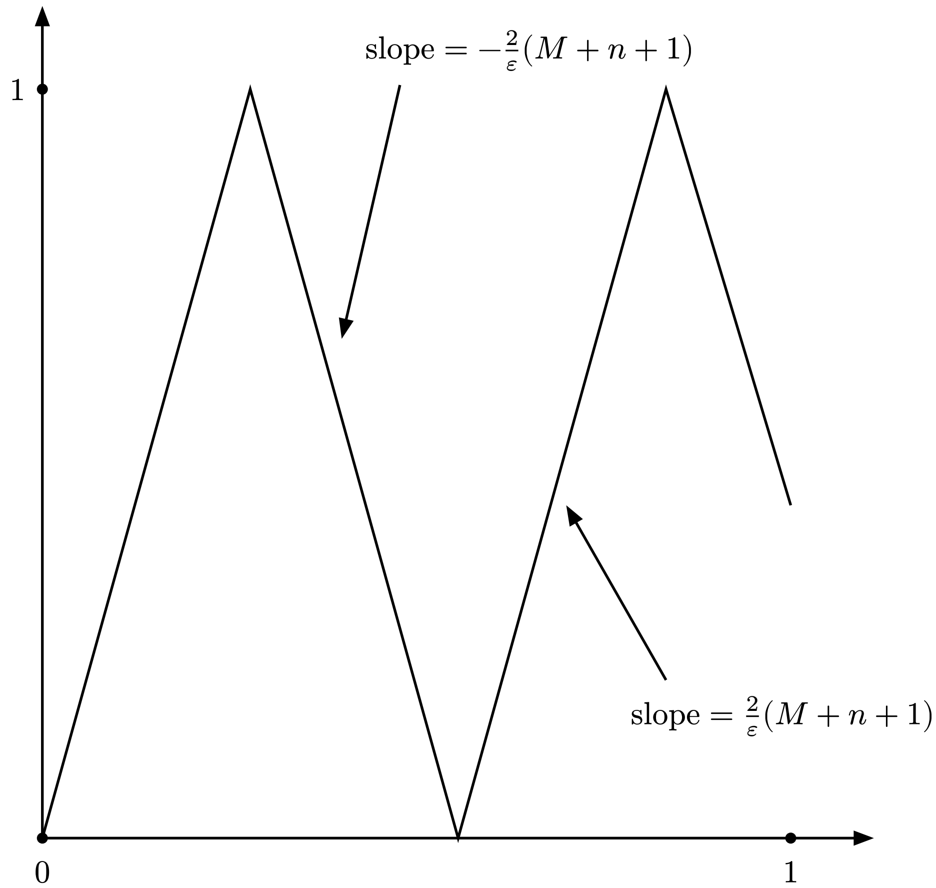
Solution.

(a) This follows from Theorem 6.7.3, which we proved in Exercise 6.7.2.

(b) Observe that

$$\|f-g\|_\infty = \left\|f-p-\tfrac{\varepsilon}{2}h\right\|_\infty \le \|f-p\|_\infty + \left\|\tfrac{\varepsilon}{2}h\right\|_\infty < \varepsilon.$$

(c) Because p is polygonal, there are points $0 = x_0 < \cdots < x_N = 1$ such that p is a line segment on $[x_{k-1}, x_k]$; for each $1 \le k \le N$, let M_k be the slope of this line segment. Define $M = \max\{|M_1|, \ldots, |M_N|\}$ and let $h \in C[0,1]$ be the sawtooth function whose slope has absolute value $\frac{2}{\varepsilon}(M+n+1)$ as in the following figure.



For any given $x \in [0,1]$, we have $x \in [x_{k-1}, x_k]$ for some $1 \le k \le N$. Note that we can always choose some $t \in [0,1]$ such that:

- $0 < |x-t| < \frac{1}{m}$;
- $t \in [x_{k-1}, x_k]$, so that x and t belong to the same line segment of p;
- x and t belong to the same line segment of h.

There are two cases. If x and t belong to a line segment of h which has slope $\frac{2}{\varepsilon}(M+n+1)$, then

$$\left|\frac{g(x)-g(t)}{x-t}\right| = \left|\frac{p(x)-p(t)}{x-t} + \frac{\varepsilon}{2}\frac{h(x)-h(t)}{x-t}\right|$$

$$= |M_k + M + n + 1| = M_k + M + n + 1 \geq n + 1 > n.$$

Similarly, if x and t belong to a line segment of h which has slope $-\frac{2}{\varepsilon}(M + n + 1)$, then

$$\left|\frac{g(x) - g(t)}{x - t}\right| = \left|\frac{p(x) - p(t)}{x - t} + \frac{\varepsilon}{2}\frac{h(x) - h(t)}{x - t}\right|$$

$$= |M_k - M - n - 1| = n + 1 + M - M_k \geq n + 1 > n.$$

To summarize: for any $x \in [0, 1]$ there exists a $t \in [0, 1]$ such that $0 < |x - t| < \frac{1}{m}$ and

$$\left|\frac{g(x) - g(t)}{x - t}\right| > n;$$

it follows that $g \notin A_{m,n}$.

We have now shown that any ε-neighbourhood of f contains some function g which does not belong to $A_{m,n}$. As f was arbitrary, this implies that each $A_{m,n}$ has empty interior. We showed in Exercise 8.2.17 that each $A_{m,n}$ is a closed set and thus each $A_{m,n}$ is nowhere-dense in $C[0, 1]$. It follows that the countable union

$$\bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{m,n}$$

is a set of first category. We showed in Exercise 8.2.16 that this union contains D and thus D is a set of first category, since any subset of a set of first category is itself a set of first category:

Lemma L.19. Let (X, d) be a metric space and suppose $A \subseteq X$ is a set of first category. If B is a subset of A then B is also a set of first category.

Proof. There is a countable collection $\{E_n : n \in \mathbf{N}\}$ of nowhere-dense sets such that $A = \bigcup_{n=1}^{\infty} E_n$. For each $n \in \mathbf{N}$, note that

$$B \cap E_n \subset E_n \quad \Rightarrow \quad \overline{B \cap E_n} \subseteq \overline{E_n} \quad \Rightarrow \quad \left(\overline{B \cap E_n}\right)^{\circ} \subseteq \left(\overline{E_n}\right)^{\circ} = \emptyset.$$

Thus $\left(\overline{B \cap E_n}\right)^{\circ} = \emptyset$, so that each $B \cap E_n$ is nowhere-dense in X. Now observe that

$$B = B \cap A = B \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (B \cap E_n).$$

This shows that B can be expressed as a countabtle union of nowhere-dense sets, i.e. B is a set of first category. □

8.3. Euler's Sum

Exercise 8.3.1. Supply the details to show that when $x = \pi/2$ the product formula in (2) is equivalent to

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right), \tag{3}$$

where the infinite product in (2) is interpreted as a limit of partial products. (Although it is not necessary for what follows, it might be useful to review the treatment of infinite products in Exercises 2.4.10 and 2.7.10.)

Solution. Let us express the product in (2) as

$$\sin(x) = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots = x\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{x}{k\pi}\right)\left(1+\frac{x}{k\pi}\right).$$

Taking $x = \frac{\pi}{2}$ gives us

$$1 = \frac{\pi}{2}\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{1}{2k}\right)\left(1+\frac{1}{2k}\right) = \frac{\pi}{2}\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k-1)(2k+1)}{2k\cdot 2k}.$$

If we let $p_n = \prod_{k=1}^{n}\frac{(2k-1)(2k+1)}{2k\cdot 2k}$, then the equation above becomes $\frac{2}{\pi} = \lim_{n\to\infty} p_n$. Note that each p_n is positive; using the continuity of $x \mapsto \frac{1}{x}$, we then have

$$\frac{\pi}{2} = \frac{1}{\lim_{n\to\infty} p_n} = \lim_{n\to\infty}\frac{1}{p_n} = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{2k\cdot 2k}{(2k-1)(2k+1)}.$$

Exercise 8.3.2. Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the integration-by-parts formula

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt\,.$$

Solution. See Exercise 7.5.6 (a).

Exercise 8.3.3.

(a) Using the simple identity $\sin^n(x) = \sin^{n-1}(x)\sin(x)$ and the previous exercise, derive the recurrence relation

$$b_n = \frac{n-1}{n} b_{n-2} \quad \text{for all } n \geq 2.$$

(b) Use this relation to generate the first three even terms and the first three odd terms of the sequence (b_n).

(c) Write a general expression for b_{2n} and b_{2n+1}.

Solution.

(a) Using integration-by-parts, we have

$$\begin{aligned}
b_n &= \int_0^{\frac{\pi}{2}} \sin^n(x)\ \mathrm{d}x \\
&= \int_0^{1} \sin^{n-1}(x)\sin(x)\ \mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-1}(x)(-\cos(x))'\ \mathrm{d}x \\
&= -\sin^{n-1}\left(\tfrac{\pi}{2}\right)\cos\left(\tfrac{\pi}{2}\right) + \sin^{n-1}(0)\cos(0) + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\cos^2(x)\ \mathrm{d}x \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)[1-\sin^2(x)]\ \mathrm{d}x \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\ \mathrm{d}x - (n-1)\int_0^{\frac{\pi}{2}} \sin^{n}(x)\ \mathrm{d}x \\
&= (n-1)b_{n-2} - (n-1)b_n.
\end{aligned}$$

The desired recurrence relation follows.

(b) Some calculations reveal that

$$b_0 = \frac{\pi}{2}, \quad b_2 = \frac{\pi}{2}\cdot\frac{1}{2}, \quad b_4 = \frac{\pi}{2}\cdot\frac{1\cdot 3}{2\cdot 4} \quad \text{and} \quad b_1 = 1, \quad b_3 = \frac{2}{1\cdot 3}, \quad b_5 = \frac{2\cdot 4}{1\cdot 3\cdot 5}.$$

(c) Simple induction arguments show that

$$b_{2n} = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \quad \text{and} \quad b_{2n+1} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

Exercise 8.3.4. Show

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1,$$

and use this fact to finish the proof of Wallis's product formula in (3).

Solution. As noted in the textbook, the sequence (b_n) is decreasing and thus

$$0 < b_{2n+1} \le b_{2n} \le b_{2n-1} \quad \Rightarrow \quad 1 \le \frac{b_{2n}}{b_{2n+1}} \le \frac{b_{2n-1}}{b_{2n+1}} \tag{$*$}$$

for each $n \in \mathbf{N}$. Using the recurrence relation from Exercise 8.3.3 (a), we have

$$\frac{b_{2n-1}}{b_{2n+1}} = 1 + \frac{1}{2n} \to 1.$$

It follows from $(*)$ and the Squeeze Theorem that $\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1$.

Now let $q_n = \prod_{k=1}^{n} \frac{2k\cdot 2k}{(2k-1)\cdot(2k+1)}$. Our goal is to show that $\frac{\pi}{2} = \lim_{n\to\infty} q_n$. Using the expressions for b_{2n} and b_{2n+1} derived in Exercise 8.3.3 (c), we find that

$$\frac{b_{2n}}{b_{2n+1}} = \frac{\pi}{2} \cdot \frac{1}{q_n} \quad \Leftrightarrow \quad q_n = \frac{\pi}{2} \cdot \frac{b_{2n+1}}{b_{2n}}.$$

It follows from the previous paragraph that $\lim_{n\to\infty} q_n = \frac{\pi}{2}$.

Exercise 8.3.5. Derive the following alternative form of Wallis's product formula:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Solution. Letting $q_n = \prod_{k=1}^{n} \frac{2k\cdot 2k}{(2k-1)\cdot(2k+1)}$, some calculations reveal that

$$q_n = \frac{1}{2} \cdot \frac{2^{4n}(n!)^4}{[(2n)!]^2 n} \cdot \frac{2n}{2n+1},$$

from which we obtain

$$\sqrt{2q_n}\sqrt{1 + \frac{1}{2n}} = \frac{2^{2n}(n!)^2}{(2n)!}.$$

The alternative formula now follows as $\sqrt{2q_n} \to \sqrt{\pi}$ by Exercise 8.3.4 and $\sqrt{1 + \frac{1}{2n}} \to 1$.

Exercise 8.3.6. Show that $1/\sqrt{1-x}$ has Taylor expansion $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1$ and

$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Solution. See Exercise 6.6.10 (a).

Exercise 8.3.7. Show that $\lim c_n = 0$ but $\sum_{n=0}^{\infty} c_n$ diverges.

Solution. If we let

$$a_n := \frac{2^{2n}(n!)^2}{(2n)!} > 0,$$

then $c_n = n^{-1/2} a_n^{-1}$. Since $a_n^{-1} \to \pi^{-1/2}$ by Exercise 8.3.5 and $n^{-1/2} \to 0$, we see that $\lim_{n\to\infty} c_n = 0$.

Because (a_n) is convergent, there is some $M > 0$ such that $a_n \leq M$ for each $n \in \mathbf{N}$, which implies that

$$c_n = \frac{a_n^{-1}}{\sqrt{n}} \geq \frac{M^{-1}}{\sqrt{n}}$$

for each $n \in \mathbf{N}$. Since $\sum_{n=1}^{\infty} M^{-1} n^{-1/2}$ is divergent by Corollary 2.4.7, the Comparison Test (Theorem 2.7.4) shows that $\sum_{n=1}^{\infty} c_n$ is divergent. It follows that $\sum_{n=0}^{\infty} c_n$ is divergent.

Exercise 8.3.8. Using the expression for $E_N(x)$ from Lagrange's Remainder Theorem, show that equation (4) is valid for all $|x| < 1/2$. What goes wrong when we try to use this method to prove (4) for all $x \in (1/2, 1)$?

Solution. See Exercise 6.6.10 (a) a small modification of that argument shows that equation (4) is valid for all $|x| \leq \frac{1}{2}$.

Exercise 8.3.9.

(a) Show

$$f(x) = f(0) + \int_0^x f'(t)\,dt.$$

(b) Now use a previous result from this section to show

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt.$$

(c) Continue in this fashion to complete the proof of the theorem.

Solution.

(a) The Fundamental Theorem of Calculus (Theorem 7.5.1 (i)) shows that

$$\int_0^x f'(t)\ \mathrm{d}t = f(x) - f(0).$$

(b) Using integration-by-parts, we have

$$\int_0^x f'(t)\ \mathrm{d}t = \int_0^x f'(t) \cdot 1\ \mathrm{d}t = xf'(x) - \int_0^x f''(t)t\ \mathrm{d}t.$$

The Fundamental Theorem of Calculus shows that $\int_0^x f''(t)\ \mathrm{d}t = f'(x) - f'(0)$; combining this with the equation above and part (a) gives

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\ \mathrm{d}t.$$

(c) Applying integration-by-parts again, we see that

$$\int_0^x f''(t)(x-t)\ \mathrm{d}t = \frac{1}{2}f''(0)x^2 + \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2\ \mathrm{d}t,$$

so that

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2\ \mathrm{d}t.$$

If we continue applying integration-by-parts, we obtain

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \cdots + \frac{1}{N!}f^{(N)}(0)x^N + \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ \mathrm{d}t \\ &= S_N(x) + \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ \mathrm{d}t. \end{aligned}$$

The desired result follows.

Exercise 8.3.10.

(a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1, 1)$, and compute $E_2(x)$ for $x = 1/2, 3/4$, and $8/9$.

(b) For a general x satisfying $-1 < x < 1$, show

$$E_2(x) = \frac{15}{16} \int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\, dt.$$

(c) Explain why the inequality

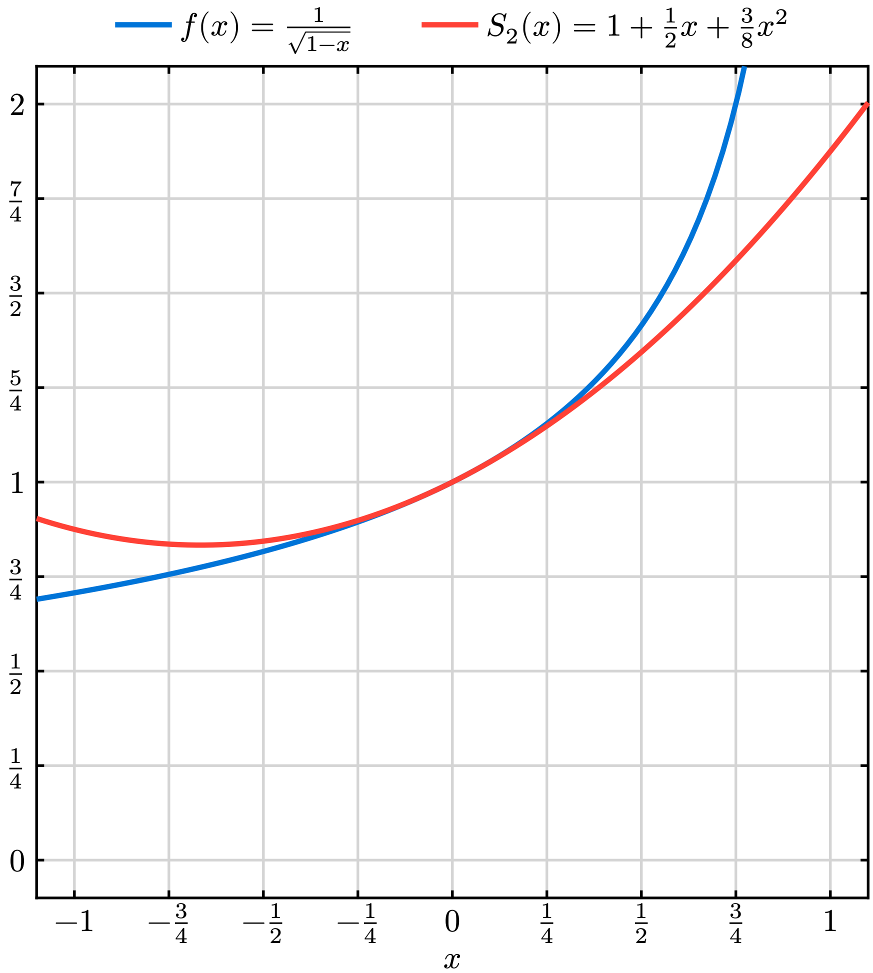
$$\left|\frac{x-t}{1-t}\right| \leq |x|$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking this overestimate is in fact larger than $|E_2(x)|$ at the three computed values from part (a).

(d) Finally, show $E_N(x) \to 0$ as $N \to \infty$ for an arbitrary $x \in (-1, 1)$.

Solution.

(a) See below for the sketch.



The errors are

$$E_2\left(\tfrac{1}{2}\right) \approx 0.0705, \quad E_2\left(\tfrac{3}{4}\right) \approx 0.4141, \quad E_2\left(\tfrac{8}{9}\right) \approx 1.2593.$$

(b) Using that

$$f^{(N)}(t) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n}(1-t)^{-n-1/2}$$

and Theorem 8.3.1, we have

$$\begin{aligned} E_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2 \, \mathrm{d}t = \frac{1}{2}\int_0^x \frac{15}{8}(1-t)^{-2-3/2}(x-t)^2 \, \mathrm{d}t \\ = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} \, \mathrm{d}t. \end{aligned}$$

(c) First suppose that $x \in [0,1)$. The inequality $0 \le t \le x < 1$ implies that

$$t \ge xt \quad \Rightarrow \quad -t \le -xt \quad \Rightarrow \quad x - t \le x - xt = x(1-t) \quad \Rightarrow \quad \frac{x-t}{1-t} \le x.$$

Since $\frac{x-t}{1-t}$ and x are both non-negative in this case, we obtain the inequality $\left|\frac{x-t}{1-t}\right| \le |x|$. Now suppose that $x \in (-1,0)$. The inequality $-1 < x \le t < 0$ implies that

$$1 \ge x \quad \Rightarrow \quad t \le xt \quad \Rightarrow \quad t - x \le xt - x = (-x)(1-t) \quad \Rightarrow \quad \frac{t-x}{1-t} \le -x.$$

Since $\frac{x-t}{1-t}$ and x are both negative in this case, we again obtain the inequality $\left|\frac{x-t}{1-t}\right| \le |x|$. Using this inequality and the expression for E_2 found in part (b), we have for $x \in [0,1)$:

$$\begin{aligned} |E_2(x)| &= \frac{15}{16}\left|\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} \, \mathrm{d}t\right| \\ &\le \frac{15}{16}\int_0^x \left|\left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\right| \mathrm{d}t \\ &\le \frac{15}{16}x^2 \int_0^x \frac{1}{(1-t)^{3/2}} \, \mathrm{d}t \\ &= \frac{15}{8}x^2 \left[\frac{1}{\sqrt{1-t}}\right]_{t=0}^{t=x} \\ &= \frac{15}{8}x^2 \left(\frac{1}{\sqrt{1-x}} - 1\right). \end{aligned}$$

Similarly for $x \in (-1,0)$:

$$\begin{aligned} |E_2(x)| = \frac{15}{16}\left|\int_x^0 \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} \, \mathrm{d}t\right| \le \frac{15}{16}x^2 \int_x^0 \frac{1}{(1-t)^{3/2}} \, \mathrm{d}t \\ = \frac{15}{8}x^2 \left[\frac{1}{\sqrt{1-t}}\right]_{t=x}^{t=0} = \frac{15}{8}x^2 \left(1 - \frac{1}{\sqrt{1-x}}\right). \end{aligned}$$

Hence for all $x \in (-1,1)$ we have the overestimate

$$|E_2(x)| \le \frac{15}{8} x^2 \left| \frac{1}{\sqrt{1-x}} - 1 \right|.$$

Denoting this overestimate by $G(x)$ and comparing with the values from part (a), we find that

$$E_2\left(\frac{1}{2}\right) \approx 0.0705 \le 0.1942 \approx G\left(\frac{1}{2}\right),$$
$$E_2\left(\frac{3}{4}\right) \approx 0.4141 \le 1.0547 \approx G\left(\frac{3}{4}\right),$$
$$E_2\left(\frac{8}{9}\right) \approx 1.2593 \le 2.9630 \approx G\left(\frac{8}{9}\right).$$

(d) Using that

$$f^{(N)}(t) = \frac{1 \cdot 3 \cdots (2N-1)}{2^N} (1-t)^{-N-1/2} = \frac{(2N)!}{2^{2N} N!} (1-t)^{-N-1/2}$$

and Theorem 8.3.1, we have for a fixed $x \in (-1,1)$:

$$\begin{aligned} E_N(x) &= \frac{1}{N!} \int_0^x f^{(N+1)}(t)(x-t)^N \, dt \\ &= \frac{(2N+2)!}{2^{2N+2} N! (N+1)!} \int_0^x \left(\frac{x-t}{1-t} \right)^N \frac{1}{(1-t)^{3/2}} \, dt \\ &= c_{N+1}(N+1) \int_0^x \left(\frac{x-t}{1-t} \right)^N \frac{1}{(1-t)^{3/2}} \, dt, \end{aligned}$$

where (c_n) was defined in Exercise 8.3.6. From this expression we can derive, as in part (c), the estimate

$$|E_N(x)| \le c_{N+1}(N+1)|x|^N \left| \frac{1}{\sqrt{1-x}} - 1 \right|.$$

Since $|x| < 1$ we have $\lim_{N \to \infty} (N+1)|x|^N = 0$ and we showed in Exercise 8.3.7 that $\lim_{N \to \infty} c_{N+1} = 0$. It now follows from the Squeeze Theorem that $\lim_{N \to \infty} |E_N(x)| = 0$.

Exercise 8.3.11. Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$, supply the justification that allows us to conclude

(5) $$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \quad \text{for all } |x| < 1.$$

Solution. Because the power series $\sum_{n=0}^{\infty} c_n x^{2n}$ converges to $(1-x^2)^{-1/2}$ on $(-1,1)$, Exercise 6.5.4 (a) shows that the power series

$$\sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$$

converges on $(-1, 1)$ and has derivatives $(1 - x^2)^{-1/2}$. As this is also the derivative of $\arcsin(x)$, Corollary 5.3.4 implies that

$$\arcsin(x) = k + \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$$

for all $x \in (-1, 1)$ and some $k \in \mathbf{R}$; taking $x = 0$ shows that $k = 0$.

Exercise 8.3.12. Our work thus far shows that the Taylor series in (5) is valid for all $|x| < 1$, but note that $\arcsin(x)$ is continuous for all $|x| \leq 1$. Carefully explain why the series in (5) converges uniformly to $\arcsin(x)$ on the closed interval $[-1, 1]$.

Solution. Observe that

$$\frac{c_n}{2n+1} = \frac{a_n^{-1}}{\sqrt{n}(2n+1)}, \quad \text{where } a_n := \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}$$

for each $n \in \mathbf{N}$. As (a_n) converges to $\sqrt{\pi}$ (see Exercise 8.3.5) and consists of strictly positive terms, there is some $L > 0$ such that $a_n \geq L$ for all $n \in \mathbf{N}$. It follows that

$$\frac{c_n}{2n+1} = \frac{a_n^{-1}}{\sqrt{n}(2n+1)} \leq \frac{L^{-1}}{\sqrt{n}(2n+1)}$$

and hence the series $\sum_{n=0}^{\infty} c_n (2n+1)^{-1}$ converges by comparison with the series $\sum_{n=1}^{\infty} n^{-3/2}$. Since each term of the series $\sum_{n=0}^{\infty} c_n (2n+1)^{-1}$ is positive, we have shown that the power series $\sum_{n=0}^{\infty} c_n (2n+1)^{-1} x^{2n+1}$ converges absolutely at $x = 1$. It follows from Theorem 6.5.2 that the convergence is uniform on $[-1, 1]$. This implies that the power series is continuous on $[-1, 1]$. Since $\arcsin(x)$ is also continuous on this interval, the function

$$D(x) = \arcsin(x) - \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$$

is continuous on $[-1, 1]$ and satisfies, by Exercise 8.3.11, $D(x) = 0$ for all $x \in (-1, 1)$. It follows from continuity that $D(-1) = D(1) = 0$ also.

Exercise 8.3.13.

(a) Show

$$\int_0^{\pi/2} \theta \, d\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1},$$

being careful to justify each step in the argument. The term b_{2n+1} refers back to our earlier work on Wallis's product.

(b) Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

and use this to finish the proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

Solution.

(a) We have

$$\int_0^{\pi/2} \theta \, \mathrm{d}\theta = \int_0^{\pi/2} \left(\sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta) \right) \mathrm{d}\theta.$$

The uniform convergence of $\sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Theorem 7.4.4, and the linearity of the integral (Theorem 7.4.2 (i)) allow us to interchange the integral with the series, obtaining

$$\int_0^{\pi/2} \theta \, \mathrm{d}\theta = \sum_{n=0}^{\infty} \left(\int_0^{\pi/2} \frac{c_n}{2n+1} \sin^{2n+1}(\theta) \right) \mathrm{d}\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1}.$$

(b) Using our formula for b_{2n+1} obtained in Exercise 8.3.3 (c), we see that $c_n b_{2n+1} = \frac{1}{2n+1}$ and hence by part (a):

$$\frac{\pi^2}{8} = \int_0^{\pi/2} \theta \, \mathrm{d}\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Now we split the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ over the odd and even positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2};$$

these manipulations are valid because these are convergent series. It follows from the above expression that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

8.4. Inventing the Factorial Function

Exercise 8.4.1. For $n \in \mathbf{N}$, let

$$n\# = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.

(b) Now prove $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbf{N}$, and revisit part (a).

Solution.

(a) We observe that $n\#$ satisfies the relation $n\# = n + (n-1)\#$ for $n \geq 2$; it seems reasonable to use this relation to extend the definition of $\#$. Thus

$$1\# = 1 + 0\# \quad \Rightarrow \quad 0\# = 1 - 1\# = 0.$$

Similarly,

$$0\# = (-1)\# = -1 + (-2)\# \quad \Rightarrow \quad (-2)\# = 1.$$

Some more calculations show that

$$1\# + (-1)\# = 1, \quad 2\# + (-2)\# = 4, \quad \text{and} \quad 3\# + (-3)\# = 9.$$

Given this, we might conjecture that $n\# + (-n)\# = n^2$ for $n \in \mathbf{N}$. Using this identity and the previous recurrence relation, we find that $\frac{1}{2}\# = \frac{1}{2} + (-\frac{1}{2})\# = \frac{3}{8}$ and thus

$$\frac{7}{2}\# = \frac{15}{2} + \frac{1}{2}\# = \frac{63}{8}.$$

(b) This is a classic proof by induction: certainly $1\# = 1 = \frac{1}{2}(1)(2)$, and if $n\# = \frac{1}{2}n(n+1)$ for some $n \in \mathbf{N}$ then

$$(n+1)\# = n + 1 + n\# = n + 1 + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

Taking $n = 0, -2$, and $\frac{7}{2}$ in this formula confirms our conjectures from part (a).

Exercise 8.4.2. Verify that the series converges absolutely for all $x \in \mathbf{R}$, that $E(x)$ is differentiable on $\mathbf{R}$, and $E'(x) = E(x)$.

Solution. For a given non-zero $x \in \mathbf{R}$, note that

$$\frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0;$$

it follows from the Ratio Test (Exercise 2.7.9) that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. Theorem 6.5.7 now implies that E is differentiable on $\mathbf{R}$ and furthermore that

$$E'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}=E(x).$$

Exercise 8.4.3.

(a) Use the results of Exercise 2.8.7 and the binomial formula to show that

$$E(x+y)=E(x)E(y)$$

for all $x,y\in\mathbf{R}$.

(b) Show that $E(0)=1, E(-x)=1/E(x)$, and $E(x)>0$ for all $x\in\mathbf{R}$.

Solution.

(a) Let $x,y\in\mathbf{R}$ be given and for each $n\geq 0$ let $a_n=\frac{y^n}{n!}$ and $b_n=\frac{x^n}{n!}$. For each $k\geq 0$, define

$$d_k=a_0b_k+\cdots+a_kb_0=\sum_{n=0}^{k}a_nb_{k-n}=\sum_{n=0}^{k}\frac{x^{k-n}y^n}{(k-n)!n!}=\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}x^{k-n}y^n=\frac{(x+y)^k}{k!}.$$

It follows that for each $N\geq 0$ we have

$$\sum_{k=0}^{N}d_k=\sum_{k=0}^{N}\frac{(x+y)^k}{k!}.$$

On one hand, $\sum_{k=0}^{N}\frac{(x+y)^k}{k!}\to E(x+y)$ as $N\to\infty$. On the other hand,

$$\sum_{k=0}^{N}d_k\to\left(\sum_{n=0}^{\infty}b_n\right)\left(\sum_{n=0}^{\infty}a_n\right)=E(x)E(y)\text{ as }N\to\infty$$

by Exercise 2.8.7. We may conclude that $E(x+y)=E(x)E(y)$.

(b) $E(0)=1$ is clear from the definition of E. Taking $y=-x$ in the identity $E(x+y)=E(x)E(y)$ shows that $E(0)=1=E(x)E(-x)$ for all $x\in\mathbf{R}$, which implies that $E(x)\neq 0$ for all $x\in\mathbf{R}$. Since E is continuous and $E(0)=1$, we must then have $E(x)>0$ for all $x\in\mathbf{R}$.

Exercise 8.4.4. Define $e=E(1)$. Show $E(n)=e^n$ and $E(m/n)=\left(\sqrt[n]{e}\right)^m$ for all $m,n\in\mathbf{Z}$.

Solution. By Exercise 8.4.3 (a) we have, for each $n\in\mathbf{N}$,

$$E(n)=E\left(\sum_{j=1}^{n}1\right)=\prod_{j=1}^{n}E(1)=\prod_{j=1}^{n}e=e^n,$$

and by Exercise 8.4.3 (b) we have $E(0)=1=e^0$. Thus the identity $E(n)=e^n$ holds for all $n\geq 0$. The identity $E(-x)=[E(x)]^{-1}$ from Exercise 8.4.3 (b) allows us to extend $E(n)=e^n$ for all $n\in\mathbf{Z}$.

For $n \in \mathbf{N}$ we have

$$e = E(1) = E\left(\sum_{j=1}^{n} \frac{1}{n}\right) = \prod_{j=1}^{n} E\left(\frac{1}{n}\right) = \left[E\left(\frac{1}{n}\right)\right]^{n}.$$

Because $E\left(\frac{1}{n}\right)$ is positive (by Exercise 8.4.3 (b)), the above equation implies that $E\left(\frac{1}{n}\right)$ is the unique positive n^{th} root of e, i.e. $E\left(\frac{1}{n}\right) = \sqrt[n]{e}$. We can now argue as in the previous paragraph to see that $E\left(\frac{m}{n}\right) = \left(\sqrt[n]{e}\right)^{m}$ for all $m \in \mathbf{Z}$ and $n \in \mathbf{N}$.

Exercise 8.4.5. Show $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 0, 1, 2, \dots$.

To get started notice that when $x \geq 0$, all the terms in (1) are positive.

Solution. We will prove the more general result that $\lim_{x\to\infty} x^n e^{-yx} = 0$ for $n \geq 0$ and $y > 0$, which will be useful later. For $x > 0$, observe that $x^n e^{-yx}$ is positive. Furthermore,

$$\begin{aligned} x^{-n}e^{yx} &= x^{-n}\left(1 + yx + \cdots + \frac{y^n x^n}{n!} + \frac{y^{n+1}x^{n+1}}{(n+1)!} + \cdots\right) \\ &= \left(\frac{1}{x^n} + \frac{y}{x^{n-1}} + \cdots + \frac{y^n}{n!} + \frac{y^{n+1}x}{(n+1)!} + \cdots\right) > \frac{y^{n+1}x}{(n+1)!}. \end{aligned}$$

Let $\varepsilon > 0$ be given and let $M = (n+1)! y^{-(n+1)} \varepsilon^{-1} > 0$. Then for $x \geq M$ we have

$$x^{-n}e^{yx} > \frac{y^{n+1}x}{(n+1)!} \geq \frac{y^{n+1}M}{(n+1)!} = \frac{1}{\varepsilon} \quad \Leftrightarrow \quad x^n e^{-yx} < \varepsilon.$$

We may conclude that $\lim_{x\to\infty} x^n e^{-yx} = 0$.

Exercise 8.4.6.

(a) Explain why we know e^x has an inverse function—let's call it $\log x$—defined on the strictly positive real numbers and satisfying

   (i) $\log(e^y) = y$ for all $y \in \mathbf{R}$ and

   (ii) $e^{\log x} = x$, for all $x > 0$.

(b) Prove $(\log x)' = 1/x$. (See Exercise 5.2.12.)

(c) Fix $y > 0$ and differentiate $\log(xy)$ with respect to x. Conclude that

$$\log(xy) = \log x + \log y \quad \text{for all } x, y > 0.$$

(d) For $t > 0$ and $n \in \mathbf{N}$, t^n has the usual interpretation $t \cdot t \cdots t$ (n times). Show that

$$t^n = e^{n \log t} \quad \text{for all } n \in \mathbf{N}. \tag{2}$$

Solution. For notation we will use either $E(x)$ or e^x depending on which is more convenient.

(a) Because $(e^x)' = e^x > 0$ (by Exercise 8.4.2 and Exercise 8.4.3 (b)), we see that E is injective (by Exercise 5.3.2). For any $y > 0$ we have

$$e^y = \left(1 + y + \frac{y^2}{2!} + \cdots\right) > y$$

and Exercise 8.4.5 shows that there is some $z < 0$ such that $e^z < y$. It follows from the Intermediate Value Theorem (Theorem 4.5.1) that there exists some $x \in (z, y)$ such that $e^x = y$. We have now shown that $E : \mathbf{R} \to (0, \infty)$ is a bijection and thus there exists an inverse function.

(b) By Exercise 5.2.12 and Exercise 8.4.2 we have

$$(\log x)' = \frac{1}{E'(\log x)} = \frac{1}{E(\log x)} = \frac{1}{x}.$$

(c) Using the chain rule and part (b), we have

$$(\log(xy))' = \frac{y}{xy} = \frac{1}{x} = (\log x)'.$$

It follows from Corollary 5.3.4 that $\log(xy) = \log x + k$ for some $k \in \mathbf{R}$. Taking $x = 1$ shows that $k = \log y$.

(d) For a given $n \in \mathbf{N}$, the identity $\log(xy) = \log x + \log y$ from part (c) shows that $n \log t = \log(t^n)$ and thus

$$e^{n \log t} = e^{\log(t^n)} = t^n.$$

Exercise 8.4.7.

(a) Show $t^{m/n} = \left(\sqrt[n]{t}\right)^m$ for all $m, n \in \mathbf{N}$.

(b) Show $\log(t^x) = x \log t$ for all $t > 0$ and $x \in \mathbf{R}$.

(c) Show t^x is differentiable on $\mathbf{R}$ and find the derivative.

Solution. For notation we will use either $E(x)$ or e^x depending on which is more convenient.

(a) Let $n \in \mathbf{N}$ be given. By Exercise 8.4.3 (a) we have

$$\left(E\left(\tfrac{1}{n} \log t\right)\right)^n = \prod_{j=1}^{n} E\left(\tfrac{1}{n} \log t\right) = E\left(\sum_{j=1}^{n} \tfrac{1}{n} \log t\right) = E(\log t) = t.$$

As $E\left(\tfrac{1}{n} \log t\right)$ is positive, it follows from the equation above that $E\left(\tfrac{1}{n} \log t\right)$ is the unique positive n^{th} root of t, i.e.

$$t^{1/n} = E\left(\frac{1}{n} \log t\right) = \sqrt[n]{t}.$$

Now let $m, n \in \mathbf{N}$ be given. By Exercise 8.4.3 (a) and the previous paragraph, we have

$$t^{m/n} = E\left(\frac{m}{n} \log t\right) = \left(E\left(\frac{1}{n} \log t\right)\right)^m = \left(\sqrt[n]{t}\right)^m.$$

(b) This is immediate from the definition of t^x:

$$\log(t^x) = \log(E(x \log t)) = x \log t.$$

(c) Using the chain rule, we find that

$$(t^x)' = (E(x \log t))' = (\log t)E'(x \log t) = (\log t)E(x \log t) = (\log t)t^x.$$

Exercise 8.4.8. Inspired by the fact that $0! = 1$ and $1! = 1$, let $h(x)$ satisfy

(i) $h(x) = 1 \quad$ for all $0 \leq x \leq 1$, and

(ii) $h(x) = xh(x-1) \quad$ for all $x \in \mathbf{R}$.

(a) Find a formula for $h(x)$ on $[1, 2]$, $[2, 3]$, and $[n, n+1]$ for arbitrary $n \in \mathbf{N}$.

(b) Now do the same for $[-1, 0]$, $[-2, -1]$, and $[-n, -n+1]$.

(c) Sketch h over the domain $[-4, 4]$.

Solution.

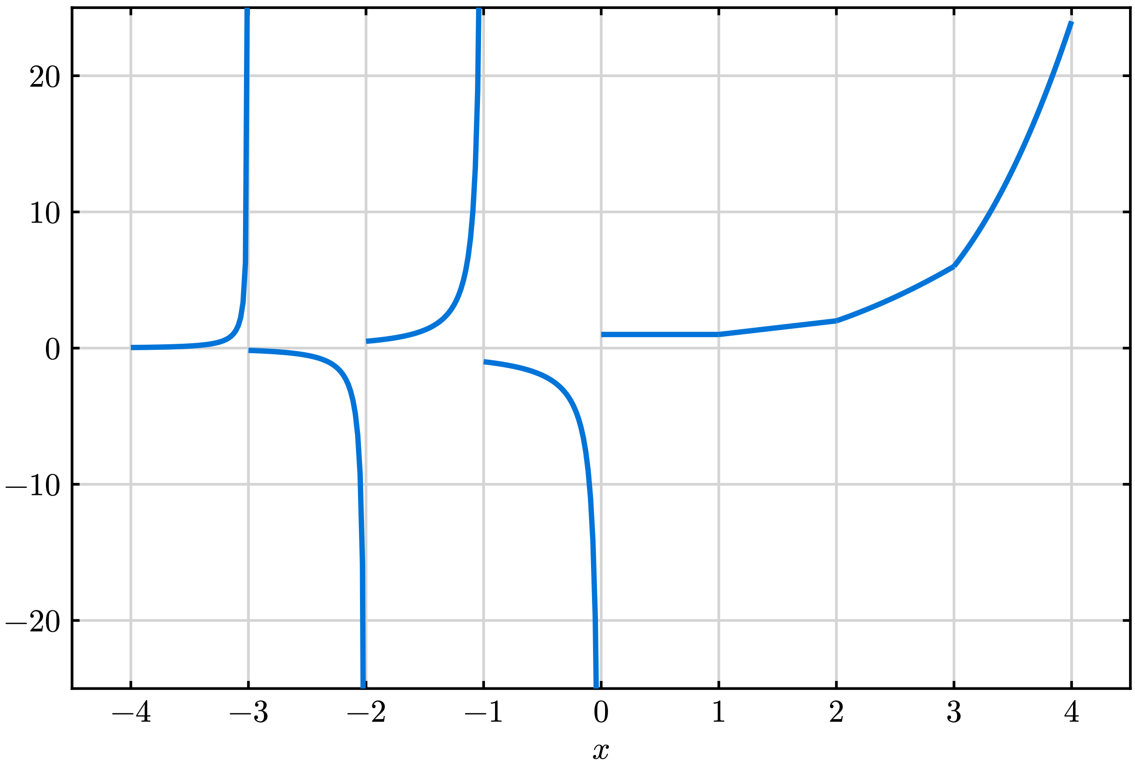
(a) On $[1, 2]$ we find that $h(x) = x$, on $[2, 3]$ we find that $h(x) = x(x-1)$, and in general we obtain $h(x) = x(x-1)\cdots(x-n+1)$ on $[n, n+1]$ for $n \in \mathbf{N}$.

(b) Replacing x with $x+1$ in (ii) we see that $h(x) = \frac{h(x+1)}{x}$ for all $x \neq 0$. Using this and (i), we find that $h(x) = \frac{1}{x}$ for $x \in [-1, 0)$. Similarly, $h(x) = \frac{1}{x(x+1)}$ for $x \in [-2, -1)$ and in general

$$h(x) = \frac{1}{x(x+1)\cdots(x+n-1)}$$

on $[-n, -n+1)$ for $n \in \mathbf{N}$.

(c) See below for the sketch.



Exercise 8.4.9.

(a) Show that the improper integral $\int_a^\infty f$ converges if and only if, for all $\varepsilon > 0$ there exists $M > a$ such that whenever $d > c \geq M$ it follows that

$$\left| \int_c^d f \right| < \varepsilon.$$

(In one direction it will be useful to consider the sequence $a_n = \int_a^{a+n} f$.)

(b) Show that if $0 \leq f \leq g$ and $\int_a^\infty g$ converges then $\int_a^\infty f$ converges.

(c) Part (a) is a Cauchy criterion, and part (b) is a comparison test. State and prove an absolute convergence test for improper integrals.

Solution.

(a) Suppose that $\int_a^\infty f$ converges to some $L \in \mathbf{R}$ and let $\varepsilon > 0$ be given. There exists an $M > a$ such that

$$b \geq M \quad \Rightarrow \quad \left| \left(\int_a^b f \right) - L \right| < \frac{\varepsilon}{2}.$$

It follows that for $d > c \geq M$ we have

$$\left| \int_c^d f \right| = \left| \left(\int_a^d f \right) - \left(\int_a^c f \right) - L + L \right| \leq \left| \left(\int_a^c f \right) - L \right| + \left| \left(\int_a^d f \right) - L \right| < \varepsilon.$$

Now suppose that

$$\text{for all } \varepsilon > 0 \text{ there exists an } M > a \text{ such that } d \geq c \geq M \text{ implies } \left| \int_c^d f \right| < \varepsilon. \qquad (*)$$

For each $n \in \mathbf{N}$ define $a_n = \int_a^{a+n} f$. Given an $\varepsilon > 0$, obtain an M from $(*)$ and let $N \in \mathbf{N}$ be such that $a + N \geq M$. If $n \geq m \geq N$ then by $(*)$ we have

$$|a_n - a_m| = \left| \int_{a+m}^{a+n} f \right| < \varepsilon.$$

Thus (a_n) is Cauchy and hence convergent, say $\lim_{n\to\infty} a_n = L$. We claim that $\int_a^\infty f = L$. To see this, let $\varepsilon > 0$ be given. By $(*)$ there is an $M > a$ such that

$$d \geq c \geq M \quad \Rightarrow \quad \left| \int_c^d f \right| < \frac{\varepsilon}{2}. \qquad (\dagger)$$

Let $N_1 \in \mathbf{N}$ be such that $a + N_1 \geq M$. Since $\lim_{n\to\infty} a_n = L$, there is an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \quad \Rightarrow \quad \left| \left(\int_a^{a+n} f \right) - L \right| < \frac{\varepsilon}{2}. \tag{‡}$$

Let $N = \max\{N_1, N_2\}$ and suppose that $b \geq a + N$. Then by (†) and (‡) we have

$$\left| \left(\int_a^b f \right) - L \right| \leq \left| \left(\int_a^{a+N} f \right) - L \right| + \left| \int_{a+N}^b f \right| < \varepsilon.$$

Our claim follows.

(b) The inequality $0 \leq f \leq g$ implies that $0 \leq \int_c^d f \leq \int_c^d g$ for any $d \geq c \geq a$. Let $\varepsilon > 0$ be given. By part (a), there is an $M > a$ such that

$$d \geq c \geq M \quad \Rightarrow \quad \left| \int_c^d g \right| = \int_c^d g < \varepsilon.$$

For such d and c we then have $\left| \int_c^d f \right| = \int_c^d f \leq \int_c^d g < \varepsilon$. It follows from part (a) that $\int_a^\infty f$ converges.

(c) We will show that if $\int_a^\infty |f|$ converges then so does $\int_a^\infty f$. For any $\varepsilon > 0$, part (a) implies that there is an $M > a$ such that $\left| \int_c^d |f| \right| = \int_c^d |f| < \varepsilon$ for any $d \geq c \geq M$. For such d and c it follows that $\left| \int_c^d f \right| \leq \int_c^d |f| < \varepsilon$ and part (a) allows us to conclude that $\int_a^\infty f$ converges.

Exercise 8.4.10.

(a) Use the properties of e^t previously discussed to show

$$\int_0^\infty e^{-t}\, dt = 1.$$

(b) Show

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\, dt, \quad \text{for all } \alpha > 0. \tag{3}$$

Solution.

(a) As $(-e^{-t})' = e^{-t}$ (by the chain rule and Exercise 8.4.2), the Fundamental Theorem of Calculus gives us

$$\lim_{b\to\infty} \int_0^b e^{-t}\ \mathrm{d}t = \lim_{b\to\infty} \left(e^0 - e^{-b}\right) = 1 - \lim_{b\to\infty} e^{-b} = 1,$$

where we have used that $e^0 = 1$ (by Exercise 8.4.3 (b)) and that $\lim_{b\to\infty} e^{-b} = 0$ (by Exercise 8.4.5).

(b) Similarly to part (a), this time using change of variables:

$$\lim_{b\to\infty}\int_0^b e^{-\alpha t}\,\mathrm{d}t = \lim_{b\to\infty}\alpha^{-1}\left(e^0 - e^{-b}\right) = \alpha^{-1}\left(1 - \lim_{b\to\infty} e^{-b}\right) = \alpha^{-1}.$$

Exercise 8.4.11.

(a) Evaluate $\int_0^b te^{-\alpha t}\,dt$ using the integration-by-parts formula from Exercise 7.5.6. The result will be an expression in α and b.

(b) Now compute $\int_0^\infty te^{-\alpha t}\,dt$ and verify equation (4).

Solution.

(a) After applying integration-by-parts and simplifying, we find that

$$\int_0^b te^{-\alpha t}\,\mathrm{d}t = \alpha^{-2} - \alpha^{-1}be^{-\alpha b} - \alpha^{-2}e^{-\alpha b}.$$

(b) Using the expression from part (a) and Exercise 8.4.5, we see that

$$\lim_{b\to\infty}\int_0^b te^{-\alpha t}\,\mathrm{d}t = \lim_{b\to\infty}\left(\alpha^{-2} - \alpha^{-1}be^{-\alpha b} - \alpha^{-2}e^{-\alpha b}\right) = \alpha^{-2}.$$

Exercise 8.4.12. Assume the function $f(x,t)$ is continuous on the rectangle

$$D = \{(x,t) : a \le x \le b, c \le t \le d\}.$$

Explain why the function

$$F(x) = \int_c^d f(x,t)\,dt$$

is properly defined for all $x \in [a,b]$.

Solution. Here is a useful lemma.

Lemma L.20. Suppose $f : D \to \mathbf{R}$ is continuous, where

$$D = \left\{(x,t) \in \mathbf{R}^2 : a \le x \le b, c \le t \le d\right\}.$$

Then for a fixed $x_0 \in [a,b]$, the function $g : [c,d] \to \mathbf{R}$ given by $g(t) = f(x_0,t)$ is continuous.

Proof. Fix $t_0 \in [c,d]$; we aim to show that g is continuous at t_0, so let $\varepsilon > 0$ be given. By assumption f is continuous at $(x_0,t_0) \in D$ and thus there is a $\delta > 0$ such that $|f(x,t) - f(x_0,t_0)| < \varepsilon$ for all $(x,t) \in D$ satisfying

$$\|(x,t)-(x_0,t_0)\| = \sqrt{(x-x_0)^2-(t-t_0)^2} < \delta.$$

Now suppose that $t \in [c,d]$ is such that $|t-t_0| < \delta$. Notice that

$$\|(x_0,t)-(x_0,t_0)\| = \sqrt{(t-t_0)^2} = |t-t_0| < \delta.$$

It follows that

$$|f(x_0,t)-f(x_0,t_0)| = |g(t)-g(t_0)| < \varepsilon$$

and hence that g is continuous at t_0, as desired. □

Solution. For a fixed $x \in [a,b]$ it is straightforward to show that the function $[c,d] \to \mathbf{R}$ given by $t \mapsto f(x,t)$ is continuous. It follows from Theorem 7.2.9 that $t \mapsto f(x,t)$ is integrable on $[c,d]$ for each $x \in [a,b]$. Thus F is properly defined.

Exercise 8.4.13. Prove Theorem 8.4.5.

Solution. Fix $x_0 \in [a,b]$; we claim that F is continuous at x_0. Let $\varepsilon > 0$ be given. Theorem 4.4.7 is easily adapted to show that f must be uniformly continuous on D and thus there exists a $\delta > 0$ such that

$$(x,t),(y,z) \in D \text{ and } \|(x,t)-(y,z)\| < \delta \quad \Rightarrow \quad |f(x,t)-f(y,z)| < \frac{\varepsilon}{d-c}.$$

Suppose that $x \in [a,b]$ is such that $|x-x_0| < \delta$. Then for any $t \in [c,d]$ we have

$$\|(x,t)-(x_0,t)\| = |x-x_0| < \delta$$

and hence $|f(x,t)-f(x_0,t)| < \varepsilon(d-c)^{-1}$. It follows that

$$|F(x)-F(x_0)| = \left|\int_c^d f(x,t)-f(x_0,t)\ \mathrm{d}t\right| \le \int_c^d |f(x,t)-f(x_0,t)|\ \mathrm{d}t \le \int_c^d \frac{\varepsilon}{d-c}\ \mathrm{d}t = \varepsilon.$$

Thus F is continuous at x_0, as claimed.

We have now shown that F is continuous on the compact set $[a,b]$. Theorem 4.4.7 now shows that F is uniformly continuous on $[a,b]$.

Exercise 8.4.14. Finish the proof of Theorem 8.4.6.

Solution. As f_x is continuous on the compact set D, it must be uniformly continuous here. Thus there exists a $\delta > 0$ such that

$$(z,s),(x,t) \in D \text{ and } |(z,s)-(x,t)| < \delta \quad \Rightarrow \quad |f_x(z,s)-f_x(x,t)| < \frac{\varepsilon}{d-c}. \qquad (*)$$

Suppose that $z \in [a,b]$ is such that $0 < |z-x| < \delta$. For a given $t \in [c,d]$, the Mean Value Theorem (Theorem 5.3.2) implies that there exists some y_t strictly between z and x, so that $|y_t - x| < |z-x| < \delta$, satisfying

$$\frac{f(z,t)-f(x,t)}{z-x} = f_x(y_t,t).$$

Notice that $\|(y_t,t)-(x,t)\| = |y_t - x| < \delta$. It follows from $(*)$ that $|f_x(y_t,t) - f_x(x,t)| < \frac{\varepsilon}{d-c}$ and hence that

$$\left|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)\ \mathrm{d}t\right| = \left|\int_c^d \frac{f(z,t)-f(x,t)}{z-x} - f_x(x,t)\ \mathrm{d}t\right|$$

$$= \left|\int_c^d f_x(y_t,t) - f_x(x,t)\ \mathrm{d}t\right| \leq \int_c^d |f_x(y_t,t)-f_x(x,t)|\ \mathrm{d}t \leq \int_c^d \frac{\varepsilon}{d-c}\ \mathrm{d}t = \varepsilon.$$

Exercise 8.4.15.

(a) Show that the improper integral $\int_0^\infty e^{-xt}$ converges uniformly to $1/x$ on the set $[1/2,\infty)$.

(b) Is the convergence uniform on $(0,\infty)$?

Solution.

(a) Let $\varepsilon > 0$ be given and let $M = \max\{-2\log(\frac{1}{2}\varepsilon), 0\}$. Then if $d \geq M$ and $x \geq \frac{1}{2}$ we have

$$\left|\frac{1}{x} - \int_0^d e^{-xt}\ \mathrm{d}t\right| = \frac{e^{-xd}}{x} \leq 2e^{-d/2} < \varepsilon;$$

we are using here that E is strictly increasing, which implies that its inverse function log is also strictly increasing.

(b) The convergence is not uniform on $(0,\infty)$. For any $M > 0$ we have

$$\left|\frac{1}{x} - \int_0^M e^{-xt}\ \mathrm{d}t\right| = \frac{e^{-Mx}}{x}.$$

Notice that $\lim_{x\to 0^+} x^{-1}e^{-Mx} = +\infty$, since $\lim_{x\to 0^+} e^{-Mx} = 1$ and $\lim_{x\to 0^+} x^{-1} = +\infty$. Thus there is an $x > 0$ such that

$$\left|\frac{1}{x} - \int_0^M e^{-xt}\ \mathrm{d}t\right| = \frac{e^{-Mx}}{x} \geq 1.$$

Exercise 8.4.16. Prove the following analogue of the Weierstrass M-Test for improper integrals: If $f(x,t)$ satisfies $|f(x,t)| \leq g(t)$ for all $x \in A$ and $\int_a^\infty g(t)\,dt$ converges, then $\int_a^\infty f(x,t)\,dt$ converges uniformly on A.

Solution. Here is a Cauchy criterion for the uniform convergence of an improper integral, an analogue of Theorem 6.4.4.

Lemma L.21. Suppose $D = \{(x,t) \in \mathbf{R}^2 : x \in A, t \geq a\}$ for some $A \subseteq \mathbf{R}$ and some $a \in \mathbf{R}$ and we have a function $f : D \to \mathbf{R}$. Then the improper integral $\int_a^\infty f(x,t)\ \mathrm{d}t$ converges uniformly to some function $F : A \to \mathbf{R}$ if and only if for every $\varepsilon > 0$ there exists an $M \geq a$ such that

$$x \in A \text{ and } c \geq b \geq M \quad \Rightarrow \quad \left| \int_b^c f(x,t)\ \mathrm{d}t \right| < \varepsilon. \tag{$*$}$$

Proof. First suppose that the improper integral $\int_a^\infty f(x,t)\ \mathrm{d}t$ converges uniformly to some function $F : A \to \mathbf{R}$ and let $\varepsilon > 0$ be given. There exists an $M \geq a$ such that

$$x \in A \text{ and } b \geq M \quad \Rightarrow \quad \left| F(x) - \int_a^b f(x,t)\ \mathrm{d}t \right| < \frac{\varepsilon}{2}.$$

Then provided $x \in A$ and $c \geq b \geq M$ we have

$$\begin{aligned} \left| \int_b^c f(x,t)\ \mathrm{d}t \right| &= \left| -F(x) + \int_a^c f(x,t)\ \mathrm{d}t + F(x) - \int_a^b f(x,t)\ \mathrm{d}t + F(x) \right| \\ &\leq \left| F(x) - \int_a^c f(x,t)\ \mathrm{d}t \right| + \left| F(x) - \int_a^b f(x,t)\ \mathrm{d}t \right| < \varepsilon. \end{aligned}$$

Now suppose that for each $\varepsilon > 0$ there exists an $M \geq a$ such that $(*)$ holds. For each $x \in A$ we may invoke Exercise 8.4.9 (a) to see that the improper integral $\int_a^\infty f(x,t)\ \mathrm{d}t$ converges; define $F(x)$ to be this value. We claim that the improper integral $\int_a^\infty f(x,t)\ \mathrm{d}t$ converges uniformly to F on A. To see this, let $\varepsilon > 0$ be given and obtain $M \geq a$ from $(*)$. If $x \in A$ and $c \geq b \geq M$ then

$$\begin{aligned} \left| F(x) - \int_a^b f(x,t)\ \mathrm{d}t \right| &= \left| F(x) - \int_a^c f(x,t)\ \mathrm{d}t + \int_b^c f(x,t)\ \mathrm{d}t \right| \\ &\leq \left| F(x) - \int_a^c f(x,t)\ \mathrm{d}t \right| + \left| \int_b^c f(x,t)\ \mathrm{d}t \right| < \left| F(x) - \int_a^c f(x,t)\ \mathrm{d}t \right| + \varepsilon. \end{aligned}$$

Notice that this inequality holds for all $c \in [b, \infty)$. Thus we can take the limit as $c \to \infty$ on both sides of the above inequality to obtain $\left| F(x) - \int_a^b f(x,t)\ \mathrm{d}t \right| \leq \varepsilon$. We may conclude that the improper integral $\int_a^\infty f(x,t)\ \mathrm{d}t$ converges uniformly to F on A. □

Returning to the exercise, let $\varepsilon > 0$ be given. By Exercise 8.4.9 (a) there exists an $M \geq a$ such that

$$x \in A \text{ and } c \geq b \geq M \quad \Rightarrow \quad \int_b^c g(t)\ \mathrm{d}t < \varepsilon.$$

It follows that for $x \in A$ and $c \geq b \geq M$ we have

$$\left| \int_b^c f(x,t)\ \mathrm{d}t \right| \leq \int_b^c |f(x,t)|\ \mathrm{d}t \leq \int_b^c g(t)\ \mathrm{d}t < \varepsilon.$$

Lemma L.21 allows us to conclude that the improper integral $\int_a^\infty f(x,t)\ \mathrm{d}t$ converges uniformly on A.

Exercise 8.4.17. Prove Theorem 8.4.8.

Solution. For each $n \in \mathbf{N}$ define $F_n : [a,b] \to \mathbf{R}$ by

$$F_n(x) = \int_c^{c+n} f(x,t)\ \mathrm{d}t.$$

By assumption f is continuous on $[a,b] \times [c, c+n]$ and so by Theorem 8.4.5 each F_n is uniformly continuous on $[a,b]$. As noted in the textbook, F_n converges uniformly to F on $[a,b]$. We may use Exercise 6.2.6 (a) to conclude that F is uniformly continuous on $[a,b]$.

Exercise 8.4.18. Prove Theorem 8.4.9.

Solution. Define $G : [a,b] \to \mathbf{R}$ and, for each $n \in \mathbf{N}$, define $F_n : [a,b] \to \mathbf{R}$ by

$$G(x) = \int_c^\infty f_x(x,t)\ \mathrm{d}t \quad \text{and} \quad F_n(x) = \int_c^{c+n} f(x,t)\ \mathrm{d}t.$$

By Theorem 8.4.6 we have $F_n'(x) = \int_c^{c+n} f_x(x,t)\ \mathrm{d}t$ and hence by assumption $F_n' \to G$ uniformly on $[a,b]$. Notice that our hypotheses imply

$$\lim_{n\to\infty} F_n(a) = \lim_{d\to\infty} \int_c^d f(a,t)\ \mathrm{d}t = F(a).$$

We may now use Theorem 6.3.3 to see that $F_n \to F$ uniformly on $[a,b]$ and furthermore that

$$F'(x) = G(x) = \int_c^\infty f_x(x,t)\ \mathrm{d}t.$$

Exercise 8.4.19.

(a) Although we verified it directly, show how to use the theorems in this section to give a second justification for the formula

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\,\mathrm{d}t, \quad \text{for all } \alpha > 0.$$

(b) Now derive the formula

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\,\mathrm{d}t, \quad \text{for all } \alpha > 0. \tag{8}$$

Solution. We will need the following results about continuous functions, the proofs of which are straightforward.

Lemma L.22. Suppose that $f, g : D \to \mathbf{R}$, where $D \subseteq \mathbf{R}^2$, are continuous functions. Then:

(i) The function $(x, y) \mapsto f(x, y)g(x, y)$ is continuous on D.

(ii) The function $(x, y) \mapsto kf(x, y)$, for some $k \in \mathbf{R}$, is continuous on D.

(iii) If $h : A \to \mathbf{R}$ is continuous, where $A \subseteq f(D) \subseteq \mathbf{R}$, then the function

$$(x, y) \mapsto h(f(x, y))$$

is continuous on D.

(a) Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(\alpha, t) = e^{-\alpha t}$. It is easy to verify that the projections $(\alpha, t) \mapsto \alpha$ and $(\alpha, t) \mapsto t$ are continuous on all of $\mathbf{R}^2$; it follows from this fact and Lemma L.22 that f is continuous on all of $\mathbf{R}^2$. Notice that $f_\alpha(\alpha, t) = -te^{-\alpha t}$ exists for all $(\alpha, t) \in \mathbf{R}^2$ we can argue as before to see that f_α is continuous on all of $\mathbf{R}^2$.

Let $0 < a < b$ be arbitrary and define $D = [a, b] \times [0, \infty)$; the previous paragraph shows that f and f_α are continuous on D. Furthermore, by Exercise 8.4.10 (b), the function $F : [a, b] \to \mathbf{R}$ given by

$$F(\alpha) = \int_0^\infty f(\alpha, t)\,\mathrm{d}t$$

is well-defined and satisfies $F(\alpha) = \alpha^{-1}$, so that $F'(\alpha) = -\alpha^{-2}$.

Now we claim that the improper integral

$$\int_0^\infty f_\alpha(\alpha, t)\,\mathrm{d}t = \int_0^\infty -te^{-\alpha t}\,\mathrm{d}t$$

converges uniformly on $[a, b]$. Notice that $|f_\alpha(\alpha, t)| = te^{-\alpha t} \le te^{-at}$ for each $\alpha \in [a, b]$ and each $t \ge 0$. Hence, by Exercise 8.4.16, it will suffice to show that the improper integral $\int_0^\infty te^{-at}\,\mathrm{d}t$ converges. (Of course, we can show directly using integration-by-parts that this integral converges to α^{-2}, as we did in Exercise 8.4.11, making this exercise redundant. However, since presumably the purpose of this exercise is to practice using the theorems and results of this section, we will proceed differently.) By Exercise 8.4.5 we have $\lim_{t\to\infty} te^{-at/2} = 0$ and so there exists an $M > 0$ such that

$$te^{-at/2} \leq 1 \quad \Leftrightarrow \quad te^{-at} \leq e^{-at/2}$$

for all $t > M$. Since $t \mapsto te^{-at}$ is continuous on $[0, M]$ it must be bounded here, say by $L \geq 0$. Thus if we define $g : [0, \infty) \to \mathbf{R}$ by

$$g(t) = \begin{cases} L & \text{if } 0 \leq t \leq M, \\ e^{-at/2} & \text{if } t > M, \end{cases}$$

then $0 \leq te^{-at} \leq g(t)$ for all $t \geq 0$. A direct calculation shows that

$$\int_0^\infty g(t)\,\mathrm{d}t = LM + \frac{2e^{-aM/2}}{a}$$

and hence by Exercise 8.4.9 (b) the improper integral $\int_0^\infty te^{-at}\,\mathrm{d}t$ also converges. We may now apply Exercise 8.4.16 to see that the improper integral $\int_0^\infty f_\alpha(\alpha, t)\,\mathrm{d}t$ converges uniformly on $[a, b]$.

We have now satisfied all the hypotheses of Theorem 8.4.9. Applying this theorem shows that

$$\frac{1}{\alpha^2} = -F'(\alpha) = -\int_0^\infty f_\alpha(\alpha, t)\,\mathrm{d}t = \int_0^\infty te^{-\alpha t}\,\mathrm{d}t$$

for all $\alpha \in [a, b]$. Since $0 < a < b$ were arbitrary, we may conclude that this formula holds for all $\alpha > 0$.

(b) Let us prove this by induction. The case $n = 0$ was handled in Exercise 8.4.10 (b) and the case $n = 1$ was handled in Exercise 8.4.11 (and also part (a) of this exercise). Suppose that the result is true for some $n \geq 0$. Let $\alpha > 0$ be given and note that, for any $b > 0$, integration-by-parts gives us

$$\int_0^b t^{n+1}e^{-\alpha t}\,\mathrm{d}t = -b^{n+1}e^{-\alpha b} + \frac{n+1}{\alpha}\int_0^b t^n e^{-\alpha t}\,\mathrm{d}t.$$

Exercise 8.4.5 shows that $\lim_{b\to\infty} b^{n+1}e^{-\alpha b} = 0$ and our induction hypothesis ensures that $\int_0^\infty t^n e^{-\alpha t}\,\mathrm{d}t = n!\alpha^{-(n+1)}$; it follows that

$$\int_0^\infty t^{n+1}e^{-\alpha t}\,\mathrm{d}t = \frac{n+1}{\alpha}\cdot\frac{n!}{\alpha^{n+1}} = \frac{(n+1)!}{\alpha^{n+2}}.$$

This completes the induction step and the proof.

Exercise 8.4.20.

(a) Show that $x!$ is an infinitely differentiable function on $(0, \infty)$ and produce a formula for the n^{th} derivative. In particular show that $(x!)'' > 0$.

(b) Use the integration-by-parts formula employed earlier to show that $x!$ satisfies the functional equation

$$(x+1)! = (x+1)x!.$$

Solution.

(a) For $n \in \mathbf{N}$ let us denote the n^{th} derivative of $x!$ by $(x!)^{(n)}$. We will prove by induction that

$$(x!)^{(n)} = \int_0^\infty (\log t)^n t^x e^{-t} \, \mathrm{d}t$$

for $x > 0$. For the base case $n = 1$, first observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(t^x e^{-t}) = (\log t) t^x e^{-t}.$$

Let $0 < a < b$ be arbitrary. We claim that the improper integral $\int_0^\infty (\log t) t^x e^{-t} \, \mathrm{d}t$ converges uniformly on $[a, b]$. To see this, note that

$$|(\log t) t^x e^{-t}| = (\log t) t^x e^{-t} \leq t^{x+1} e^{-t} \leq t^{b+1} e^{-t}$$

for $x \in [a, b]$ and $t \geq 1$. Note further that

$$|(\log t) t^x e^{-t}| = |\log t| t^x e^{-t} \leq |\log t| t^b$$

for $x \in [a, b]$ and $0 < t < 1$. Since

$$\lim_{t \to 0^+} |\log t| t^b = 0,$$

which can be seen using L'Hôpital's rule, there exists an $M > 0$ such that $|\log t| t^b \leq M$ for all $x \in [a, b]$ and $0 < t < 1$. Thus, if we define

$$g(t) = \begin{cases} M & \text{if } 0 < t < 1, \\ t^{b+1} e^{-t} & \text{if } t \geq 1, \end{cases}$$

then $|(\log t) t^x e^{-t}| \leq g(t)$. It is straightforward to show that $\int_0^\infty g(t) \, \mathrm{d}t$ converges and so it follows from Exercise 8.4.16 that $\int_0^\infty (\log t) t^x e^{-t} \, \mathrm{d}t$ converges uniformly on $[a, b]$. We can now use Theorem 8.4.9 to see that

$$(x!)' = \int_0^\infty (\log t) t^x e^{-t} \, \mathrm{d}t$$

for $x \in [a, b]$. Since $0 < a < b$ were arbitrary, we see that this formula holds for all $x > 0$.

The induction step is essentially identical to the base case; note that

$$\frac{\mathrm{d}}{\mathrm{d}x}((\log t)^n t^x e^{-t}) = (\log t)^{n+1} t^x e^{-t}.$$

For arbitrary $0 < a < b$, we can again bound $\left|(\log t)^{n+1} t^x e^{-t}\right|$ by

$$g(t) = \begin{cases} M & \text{if } 0 < t < 1, \\ t^{b+n+1} e^{-t} & \text{if } t \geq 1, \end{cases}$$

where $M > 0$ is some bound on $\left|(\log t)^{n+1} t^x e^{-t}\right|$ for $x \in [a, b]$ and $0 < t < 1$; the existence of this M follows since

$$\lim_{t \to 0^+} |\log t|^{n+1} t^b = 0,$$

which can be seen by repeated applications of L'Hôpital's rule. Since $\int_0^\infty g(t)\ \mathrm{d}t$ converges, Exercise 8.4.16 implies that the improper integral $\int_0^\infty (\log t)^{n+1} t^x e^{-t}\ \mathrm{d}t$ converges uniformly on $[a, b]$ and hence by Theorem 8.4.9 we have

$$(x!)^{(n+1)} = \frac{\mathrm{d}}{\mathrm{d}x}(x!)^{(n)} = \int_0^\infty (\log t)^{n+1} t^x e^{-t}\ \mathrm{d}t$$

for all $x \in [a, b]$. Since $0 < a < b$ were arbitrary, the formula holds for all $x > 0$. This completes the induction step and the proof.

In particular, we have

$$(x!)'' = \int_0^\infty (\log t)^2 t^x e^{-t}\ \mathrm{d}t.$$

The integrand $(\log t)^2 t^x e^{-t}$ is strictly positive for all $x > 0$ and all $t > 1$. Thus $(x!)'' > 0$.

(b) For any $b > 0$, integration-by-parts gives

$$\int_0^b t^{x+1} e^{-t}\ \mathrm{d}t = -b^{x+1} e^{-b} + (x+1) \int_0^b t^x e^{-t}\ \mathrm{d}t,$$

which converges to $(x+1)x!$ as $b \to \infty$.

Exercise 8.4.21.

(a) Use the convexity of $\log(f(x))$ and the three intervals $[n-1,n],[n,n+x]$, and $[n,n+1]$ to show

$$x\log(n) \le \log(f(n+x)) - \log(n!) \le x\log(n+1).$$

(b) Show $\log(f(n+x)) = \log(f(x)) + \log((x+1)(x+2)\cdots(x+n))$.

(c) Now establish that

$$0 \le \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \le x\log\left(1+\frac{1}{n}\right).$$

(d) Conclude that

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}, \quad \text{for all } x \in (0,1].$$

(e) Finally, show that the conclusion in (d) holds for all $x \ge 0$.

Solution.

(a) First consider the intervals $[n-1,n]$ and $[n,n+x]$. Using the fact about convex functions mentioned previously in the textbook, we find the inequality

$$\log(f(n)) - \log(f(n-1)) \le \frac{\log(f(n+x)) - \log(f(n))}{x}.$$

Since $f(n) = n!$ and $\log(a) - \log(b) = \log(\frac{a}{b})$, we have

$$\log(f(n)) - \log(f(n-1)) = \log(n!) - \log((n-1)!) = \log\left(\frac{n!}{(n-1)!}\right) = \log(n).$$

Thus we obtain $x\log(n) \le \log(f(n+x)) - \log(n!)$. A similar argument with the intervals $[n,n+x]$ and $[n,n+1]$ (remembering that $x \le 1$) gives us the other desired inequality.

(b) Property (ii) implies that

$$f(x+n) = f(x)(x+1)(x+2)\cdots(x+n).$$

Now we can use that $\log(ab) = \log(a) + \log(b)$ to obtain the desired equality.

(c) Part (a) gives us

$$0 \le \log(f(n+x)) - \log(n!) - x\log(n) \le x\log(n+1) - x\log(n).$$

Part (b) and the usual properties of logarithms imply that

$$\begin{aligned}\log(f(n+x)) - \log(n!) - x\log(n) &= \log(f(x)) + \log((x+1)\cdots(x+n)) - \log(n^x n!) \\ &= \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)\cdots(x+n)}\right).\end{aligned}$$

Similarly,

$$x\log(n+1) - x\log(n) = x(\log(n+1) - \log(n)) = x\log\left(\frac{n+1}{n}\right) = x\log\left(1+\frac{1}{n}\right).$$

Combining these gives the desired result.

(d) Since $\log\left(1+\frac{1}{n}\right) \to 0$, the Squeeze Theorem and part (c) imply that

$$\log(f(x)) = \lim_{n\to\infty} a_n \quad \text{where } a_n = \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right)$$

for each $x \in (0,1]$. Since the exponential function is continuous everywhere, the above equation implies that

$$f(x) = e^{\lim a_n} = \lim_{n\to\infty} e^{a_n} = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}$$

for each $x \in (0,1]$.

(e) For $x = 0$ we have

$$\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)} = \frac{n^0 n!}{n!} = 1 = f(0).$$

For $x > 0$, let $m \in \mathbf{N}$ be such that $x \in (0,m]$. By repeating our previous argument with the intervals $[n-1,n], [n,n+x]$ and $[n,n+m]$, we arrive at the inequality

$$0 \le \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \le \frac{x}{m}\log\left(\frac{(n+m)!}{n!n^m}\right).$$

Notice that

$$\frac{(n+m)!}{n!n^m} = \frac{(n+m)(n+m-1)\cdots(n+1)}{n^m} = \left(1+\frac{m}{n}\right)\left(1+\frac{m-1}{n}\right)\cdots\left(1+\frac{1}{n}\right).$$

Since each of the m terms in parentheses on the right-hand side converges to 1, we see that $\lim_{n\to\infty} \frac{(n+m)!}{n!n^m} = 1$ and thus

$$\lim_{n\to\infty} \frac{x}{m}\log\left(\frac{(n+m)!}{n!n^m}\right) = 0.$$

We can now argue as in part (d) using the Squeeze Theorem to see that

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}.$$

Exercise 8.4.22.

(a) Where does $g(x) = \frac{x}{x!(-x)!}$ equal zero? What other familiar function has the same set of roots?

(b) The function e^{-x^2} provides the raw material for the all-important Gaussian bell curve from probability, where it is known that $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$. Use this fact (and some standard integration techniques) to evaluate $(1/2)!$.

(c) Now use (a) and (b) to conjecture a striking relationship between the factorial function and a well-known function from trigonometry.

Solution.

(a) We are taking $\frac{1}{x!}$ to be zero when $x = -1, -2, -3, \ldots$ and thus g is zero at each integer. The function $\sin(\pi x)$ has the same set of roots.

(b) For any $b > 0$, standard integration techniques give us

$$\int_0^b \sqrt{t}e^{-t}\,\mathrm{d}t = \int_0^{\sqrt{b}} 2u^2 e^{-u^2}\,\mathrm{d}u = -\sqrt{b}e^{-b} + \int_0^{\sqrt{b}} e^{-u^2}\,\mathrm{d}u,$$

which, given that $\int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$, converges to $\frac{1}{2}\sqrt{\pi}$ as $b \to \infty$. Thus

$$(1/2)! = \int_0^\infty \sqrt{t}e^{-t}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

(c) We conjecture that $\frac{x}{x!(-x)!} = k\sin(\pi x)$ for some $k \in \mathbf{R}$. Taking $x = \frac{1}{2}$ gives us

$$k = \frac{1/2}{(1/2)!(-1/2)!}.$$

Using part (b) and the identity $(1/2)! = (1/2)(-1/2)!$, we find that $k = \pi^{-1}$.

Exercise 8.4.23. As a parting shot, use the value for $(1/2)!$ and the Gauss product formula in equation (9) to derive the famous product for π discovered by John Wallis in the 1650s:

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

Solution. Taking $x = 1/2$ in equation (9) gives

$$(1/2)! = \frac{\sqrt{\pi}}{2} = \lim_{n\to\infty} \frac{\sqrt{n}(n!)}{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2n+2}{2}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}2^n(n!)}{3\cdot 5\cdots(2n+1)} = \lim_{n\to\infty} \frac{\sqrt{n}\cdot 2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}.$$

Squaring both sides of this equality, using the continuity of $x \mapsto x^2$, and multiplying through by $\frac{\pi}{2}$ gives us

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{3\cdot 3}\right)\left(\frac{4\cdot 4}{5\cdot 5}\right)\cdots\left(\frac{2n\cdot 2n}{(2n+1)(2n+1)}\right)(2n)$$

$$= \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right)\left(\frac{2n}{2n+1}\right).$$

Since $\lim_{n\to\infty} \frac{2n}{2n+1} = 1$, it must be the case that

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

8.5. Fourier Series

Exercise 8.5.1.

(a) Verify that

$$u(x,t) = b_n \sin(nx)\cos(nt)$$

satisfies equations (1), (2), and (3) for any choice of $n \in \mathbf{N}$ and $b_n \in \mathbf{R}$. What goes wrong if $n \notin \mathbf{N}$?

(b) Explain why any finite sum of functions of the form given in part (a) would also satisfy (1), (2), and (3). (Incidentally, it is possible to hear the different solutions in (a) for values of n up to 4 or 5 by isolating the harmonics on a well-made stringed instrument.)

Solution.

(a) Let $n \in \mathbf{N}$ and $b_n \in \mathbf{R}$ be given. Calculations show that

$$\frac{\partial u}{\partial t} - nb_n \sin(nx)\sin(nt), \quad \frac{\partial^2 u}{\partial t^2} - n^2 b_n \sin(nx)\cos(nt),$$

$$\text{and} \quad \frac{\partial^2 u}{\partial x^2} = -n^2 b_n \sin(nx)\cos(nt).$$

It is then clear that u satisfies equations (1), (2), and (3). If $n \notin \mathbf{N}$ then it may no longer be the case that u satisfies equations (2) and (3).

(b) If u and v both satisfy equations (1), (2), and (3), then observe that

$$\frac{\partial^2}{\partial x^2}(u+v) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2}{\partial t^2}(u+v).$$

Thus $u+v$ also satisfies equation (1). Furthermore,

$$u(0,t) + v(0,t) = 0 \quad \text{and} \quad u(\pi,t) + v(\pi,t) = 0$$

for all $t \geq 0$, so that $u+v$ also satisfies equation (2). Finally,

$$\frac{\partial}{\partial t}[u+v](x,0) = \frac{\partial u}{\partial t}(x,0) + \frac{\partial v}{\partial t}(x,0) = 0$$

for all $x \in [0,\pi]$, so that $u+v$ also satisfies equation (3).

Exercise 8.5.2. Using trigonometric identities when necessary, verify the following integrals.

(a) For all $n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos(nx)\ \mathrm{d}x = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\ \mathrm{d}x = 0.$$

(b) For all $n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\ \mathrm{d}x = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)\ \mathrm{d}x = \pi.$$

(c) For all $m, n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ \mathrm{d}x = 0.$$

For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ \mathrm{d}x = 0.$$

Solution.

(a) Let $n \in \mathbf{N}$ be given. A calculation shows that

$$\int_{-\pi}^{\pi} \cos(nx)\ \mathrm{d}x = \frac{[\sin(nx)]_{x=-\pi}^{x=\pi}}{n} = 0.$$

Notice that $\sin(nx)$ is an odd function. An odd function integrated over an interval of the form $[-a, a]$ is necessarily zero and hence

$$\int_{-\pi}^{\pi} \sin(nx)\ \mathrm{d}x = 0.$$

(b) Let $n \in \mathbf{N}$ be given. Using the identity $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, we calculate

$$\int_{-\pi}^{\pi} \cos^2(nx)\ \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1 + \cos(2nx)}{2}\ \mathrm{d}x = \left[\frac{x}{2} + \frac{\sin(2nx)}{4n}\right]_{x=-\pi}^{x=\pi} = \pi.$$

Similarly, using the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, we calculate

$$\int_{-\pi}^{\pi} \sin^2(nx)\ \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1 - \cos(2nx)}{2}\ \mathrm{d}x = \left[\frac{x}{2} - \frac{\sin(2nx)}{4n}\right]_{x=-\pi}^{x=\pi} = \pi.$$

(c) For any $m, n \in \mathbf{N}$, notice that $\cos(mx)\sin(nx)$ is the product of an even function and an odd function and hence is itself an odd function; it follows that

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ \mathrm{d}x = 0.$$

Now suppose $m \neq n$. Using the identity $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$, we calculate

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\cos((m-n)x) + \cos((m+n)x)}{2}\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} + \frac{\sin((m+n)x)}{m+n}\right]_{x=-\pi}^{x=\pi} = 0.\end{aligned}$$

Using the identity $\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$, we calculate

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2}\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_{x=-\pi}^{x=\pi} = 0.\end{aligned}$$

Exercise 8.5.3. Derive the formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \quad \text{and} \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x \tag{10}$$

for all $m \geq 1$.

Solution. Let $m \geq 1$ be given. Multiply both sides of equation (6) by $\cos(mx)$ and integrate over $[-\pi, \pi]$ to obtain

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi}\left(a_0\cos(mx) + \sum_{n=1}^{\infty} a_n\cos(mx)\cos(nx) + b_n\cos(mx)\sin(nx)\right)\mathrm{d}x.$$

Now, assuming we are justified in doing so, we swap the integral with the sum and use Exercise 8.5.2 to find that

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \pi a_m.$$

We can find b_m similarly, multiplying equation (6) by $\sin(mx)$ instead.

Exercise 8.5.4.

(a) Referring to the previous example, explain why we can be sure that the convergence of the partial sums to $f(x)$ is *not* uniform on any interval containing 0.

(b) Repeat the computations of Example 8.5.1 for the function $g(x) = |x|$ and examine graphs for some partial sums. This time, make use of the fact that g is even ($g(x) = g(-x)$) to simplify the calculations. By just looking at the coefficients, how do we know this series converges uniformly to something?

(c) Use graphs to collect some empirical evidence regarding the question of term-by-term differentiation in our two examples to this point. Is it possible to conclude convergence or divergence of either differentiated series by looking at the resulting coefficients? Theorem 6.4.3 is about the legitimacy of term-by-term differentiation. Can it be applied to either of these examples?

Solution.

(a) Each partial sum S_N is continuous at 0, whereas f is not. It follows from Theorem 6.2.6 that the convergence cannot be uniform.

(b) The fact that g is even implies that each b_n is zero. We calculate

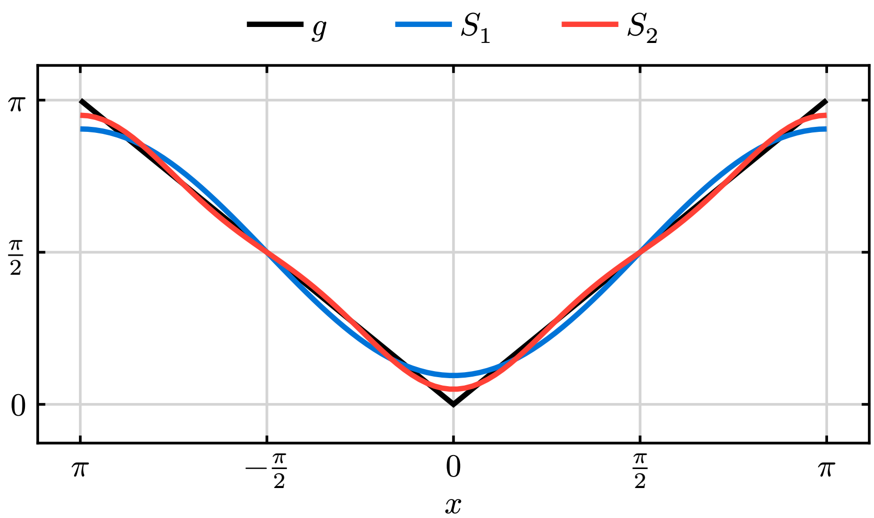
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \, \mathrm{d}x = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx) \, \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx) \, \mathrm{d}x = \begin{cases} -\frac{4}{n^2\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus the Fourier series for g is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2};$$

see below for a graph of g, S_1, and S_2 over $[-\pi, \pi]$.



Notice that

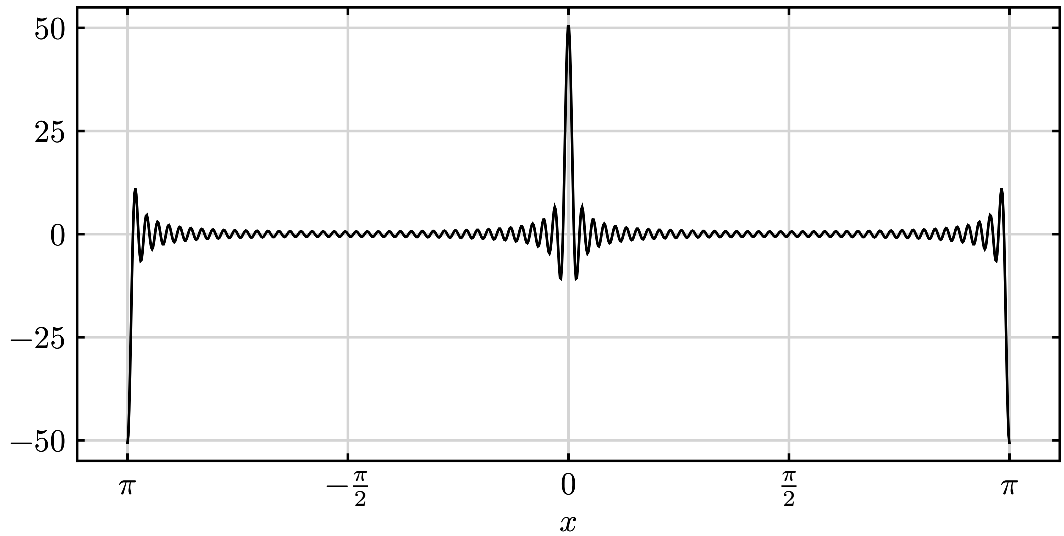
$$\left|\frac{\cos(2n-1)x}{(2n-1)^2}\right| \le \frac{1}{(2n-1)^2}$$

for each $n \in \mathbf{N}$ and $x \in [-\pi, \pi]$. It follows from the Weierstrass M-Test that the series converges uniformly.

(c) For the function f from Example 8.5.1, notice that f is not differentiable at $x = 0$ or at $x = \pm\pi$, but satisfies $f'(x) = 0$ for all $x \in (-\pi, 0) \cup (0, \pi)$. The term-by-term differentiated series is

$$\frac{4}{\pi}\sum_{n=0}^{\infty} \cos((2n+1)x).$$

See below for a graph of $\frac{4}{\pi}\sum_{n=0}^{40} \cos((2n+1)x)$ over $[-\pi, \pi]$.



The series clearly diverges for $x = 0$ and $x = \pm\pi$; this behaviour is reflected in the graph. However, based on the graph we might naively believe that the series is converging to $f'(x) = 0$ for all $x \in (-\pi, 0) \cup (0, \pi)$. In fact, this series converges if and only if $x = m\pi + \frac{\pi}{2}$ for some $m \in \mathbf{Z}$. If we put the term-by-term differentiated series in the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

then the coefficients are given by

$$b_n = 0, \quad a_0 = 0, \quad \text{and} \quad a_n = \begin{cases} \frac{4}{\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which certainly do not allow us to conclude convergence of the term-by-term differentiated series. Furthermore, we cannot use Theorem 6.4.3 since the term-by-term differentiated series does not even converge pointwise, let alone uniformly.

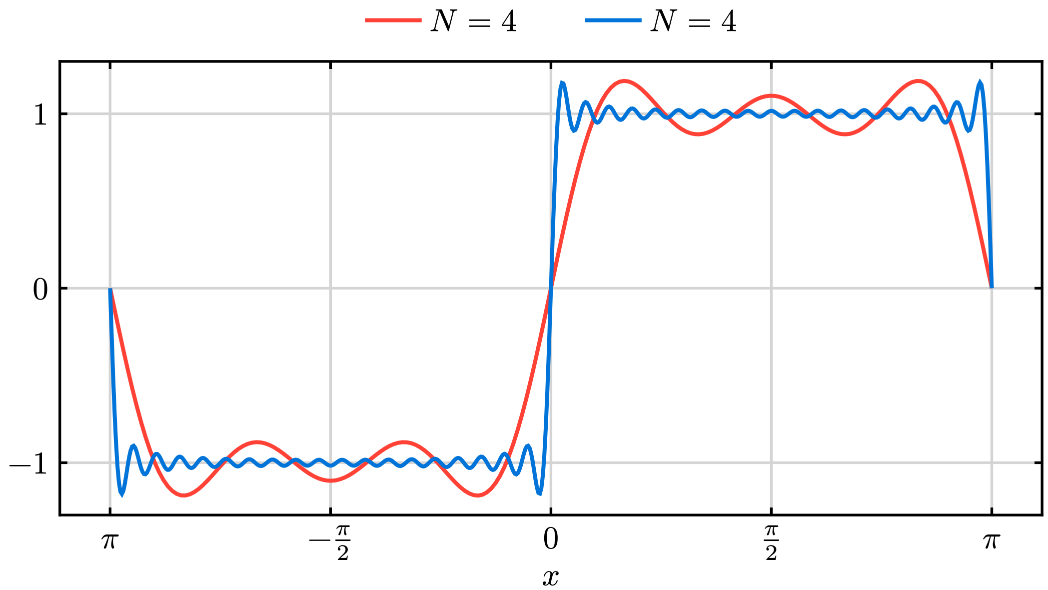
For the function g from part (b), notice that g is not differentiable at $x = 0$ or at $x = \pm\pi$, but satisfies

$$g'(x) = \begin{cases} -1 & \text{if} -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

Notice the similarity to f; indeed, the term-by-term differentiated series is identical to the Fourier series for f:

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}.$$

See below for a graph of $\frac{4}{\pi} \sum_{n=1}^{N} \frac{\sin((2n-1)x)}{2n-1}$ over $[-\pi, \pi]$ for $N = 4$ and $N = 20$.



If we put the term-by-term differentiated series in the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

then the coefficients are given by

$$a_n = 0 \quad \text{and} \quad b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which certainly do not allow us to conclude convergence of the term-by-term differentiated series. To use Theorem 6.4.3, we would have to show that the term-by-term differentiated series converges uniformly. At this stage, it is not clear how to do so.

Exercise 8.5.5. Explain why h is uniformly continuous on $\mathbf{R}$.

Solution. By assumption h is continuous on the compact set $[-\pi, \pi]$ and thus, by Theorem 4.4.7, h is uniformly continuous on $[-\pi, \pi]$. This is sufficient to show that h is uniformly continuous on $\mathbf{R}$ since, by the 2π-periodicity of h, for any $x, y \in \mathbf{R}$ there exist integers m, n such that $x + 2m\pi \in [-\pi, \pi]$ and $y + 2n\pi \in [-\pi, \pi]$.

Exercise 8.5.6. Show that $\left|\int_a^b h(x)\sin(nx)\,dx\right| < \varepsilon/n$, and use this fact to complete the proof.

Solution. Let us slightly modify the start of the proof by instead choosing a $\delta > 0$ such that $|h(x) - h(y)| < \varepsilon(2\pi)^{-1}$ whenever $|x - y| < \delta$. For $x \in [a, b]$, define $g(x) = h(x) - h\big(\frac{1}{2}(a+b)\big)$ and note that $|g(x)| < \varepsilon(2\pi)^{-1}$ since

$$\left|x - \frac{a+b}{2}\right| \leq \frac{b-a}{2} = \frac{\pi}{n} < \delta.$$

By $\frac{2\pi}{n}$-periodicity we have

$$\int_a^b \sin(nx)\ \mathrm{d}x = \int_{-\pi/n}^{\pi/n} \sin(nx)\ \mathrm{d}x = 0.$$

Since $|\sin(nx)| \leq 1$ for all $x \in \mathbf{R}$ it follows that

$$\begin{aligned}\left|\int_a^b h(x)\sin(nx)\ \mathrm{d}x\right| &\leq \left|h\left(\frac{a+b}{2}\right)\int_a^b \sin(nx)\ \mathrm{d}x\right| + \left|\int_a^b g(x)\sin(nx)\ \mathrm{d}x\right| \\ &\leq \int_a^b |g(x)||\sin(nx)|\ \mathrm{d}x \leq \int_a^b \frac{\varepsilon}{2\pi}\ \mathrm{d}x = \frac{\varepsilon}{2\pi}\cdot\frac{2\pi}{n} = \frac{\varepsilon}{n}.\end{aligned}$$

Now let $x_0 < x_1 < \cdots < x_n$ be the evenly spaced partition of $[-\pi, \pi]$ such that each subinterval has length $\frac{2\pi}{n}$. Then

$$\left|\int_{-\pi}^{\pi} h(x)\sin(nx)\ \mathrm{d}x\right| = \left|\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} h(x)\sin(nx)\ \mathrm{d}x\right| \leq \sum_{j=1}^{n}\left|\int_{x_{j-1}}^{x_j} h(x)\sin(nx)\ \mathrm{d}x\right| < \sum_{j=1}^{n}\frac{\varepsilon}{n} = \varepsilon.$$

Thus $\int_{-\pi}^{\pi} h(x)\sin(nx)\ \mathrm{d}x \to 0$ and by repeating this argument with sin replaced by cos, we can show that $\int_{-\pi}^{\pi} h(x)\cos(nx)\ \mathrm{d}x \to 0$.

Exercise 8.5.7.

(a) First, argue why the integral involving $q_x(u)$ tends to zero as $N \to \infty$.

(b) The first integral is a little more subtle because the function $p_x(u)$ has the $\sin(u/2)$ term in the denominator. Use the fact that f is differentiable at x (and a familiar limit from calculus) to prove that the first integral goes to zero as well.

Solution.

(a) The continuity of f implies the continuity of q_x and thus by the Riemann-Lebesgue Lemma (Theorem 8.5.2) we have

$$\int_{-\pi}^{\pi} q_x(u)\cos(Nu)\ \mathrm{d}u \to 0 \text{ as } N \to \infty.$$

(b) The continuity of p_x on $(-\pi, 0) \cup (0, \pi]$ follows as f, sin, and cos are continuous everywhere and sin is non-zero on $\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Strictly speaking, p_x is not defined at $u = 0$. We claim that defining $p_x(0) = 2f'(x)$ results in p_x also being continuous at zero. Observe that for $u \neq 0$:

$$\frac{f(u+x) - f(x)}{\sin(u/2)} = 2 \cdot \frac{f(u+x) - f(x)}{u} \cdot \frac{u/2}{\sin(u/2)} \to 2f'(x) \text{ as } u \to 0,$$

where we have used that f is differentiable at x and also that $\lim_{u\to\infty} \frac{u}{\sin(u)} = 1$. Thus p_x is continuous on $(-\pi, \pi]$ and so we may again use the Riemann-Lebesgue Lemma to conclude that

$$\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\ \mathrm{d}u \to 0 \text{ as } N \to \infty.$$

Exercise 8.5.8. Prove that if a sequence of real numbers (x_n) converges, then the arithmetic means

$$y_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

also converge to the same limit. Give an example to show that it is not possible for the sequence of means (y_n) to converge even if the original sequence (x_n) does not.

Solution. Suppose that $\lim x_n = x$ and let $\varepsilon > 0$ be given. There exists an $N_1 \in \mathbf{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ whenever $n \geq N_1$. Choose $N_2 \in \mathbf{N}$ such that

$$\frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{N_2} < \frac{\varepsilon}{2}$$

and suppose that $n > \max\{N_1, N_2\}$. It follows that

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 + \cdots + x_{N_1+1} + \cdots + x_n}{n} - \frac{nx}{n} \right| \\
&= \left| \frac{(x_1 - x) + \cdots + (x_{N_1} - x)}{n} + \frac{(x_{N_1+1} - x) + \cdots + (x_n - x)}{n} \right| \\
&\leq \frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} + \frac{|x_{N_1+1} - x| + \cdots + |x_n - x|}{n} \\
&< \frac{\varepsilon}{2} + \frac{n - N_1}{n} \cdot \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Thus $\lim y_n = x$. For an example where (x_n) does not converge but (y_n) does, let $(x_n) = (-1)^n$. Then

$$y_n = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which converges to zero.

Exercise 8.5.9. Use the previous identity to show that

$$\frac{1/2 + D_1(\theta) + D_2(\theta) + \cdots + D_N(\theta)}{N+1} = \frac{1}{2(N+1)}\left[\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right]^2.$$

Solution. It will suffice to show that

$$1 + 2D_1(\theta) + \cdots + 2D_N(\theta) = \frac{\sin^2\left((N+1)\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}.$$

Indeed, using the identities

$$\sin(\alpha)\sin(\theta) = \frac{\cos(\alpha-\theta) - \cos(\alpha+\theta)}{2} \quad \text{and} \quad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2},$$

we find that

$$\begin{aligned}
2\sum_{k=0}^{N} D_k(\theta) &= \sum_{k=0}^{N} \frac{\sin\left((k+\frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} \sum_{k=0}^{N} \left[\sin\left((k+\tfrac{1}{2})\theta\right)\sin\left(\tfrac{\theta}{2}\right)\right] \\
&= \frac{1}{2\sin^2\left(\frac{\theta}{2}\right)} \sum_{k=0}^{N} [\cos(k\theta) - \cos((k+1)\theta)] \\
&= \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} \cdot \frac{1-\cos((N+1)\theta)}{2} \\
&= \frac{\sin^2\left((N+1)\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

Exercise 8.5.10.

(a) Show that

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\,du.$$

(b) Graph the function $F_N(u)$ for several values of N. Where is F_N large, and where is it close to zero? Compare this function to the Dirichlet kernel $D_N(u)$. Now, prove that $F_N \to 0$ uniformly on any set of the form $\{u : |u| \geq \delta\}$, where $\delta > 0$ is fixed (and u is restricted to the interval $(-\pi, \pi]$).

(c) Prove that $\int_{-\pi}^{\pi} F_N(u)\,du = \pi$.

(d) To finish the proof of Fejér's Theorem, first choose a $\delta > 0$ so that

$$|u| < \delta \quad \text{implies} \quad |f(x+u) - f(x)| < \varepsilon.$$

Set up a single integral that represents the difference $\sigma_N(x) - f(x)$ and divide this integral into sets where $|u| \leq \delta$ and $|u| \geq \delta$. Explain why it is possible to make each of these integrals sufficiently small, independently of the choice of x.

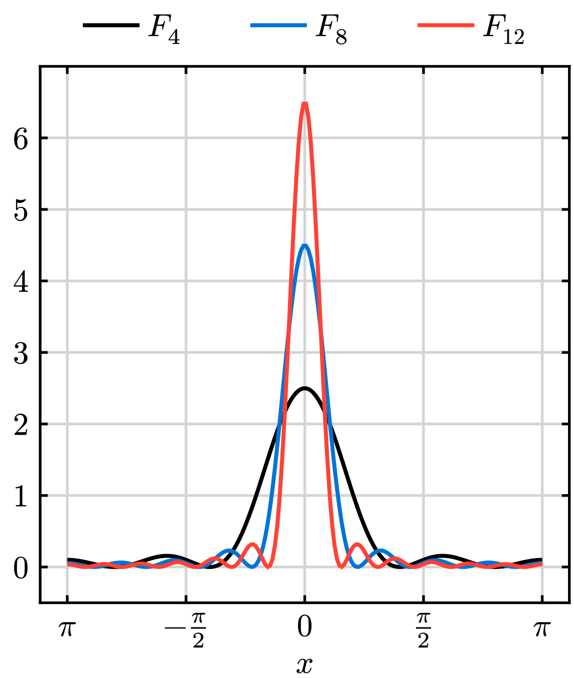
Solution.

(a) Using the expression for $S_n(x)$ derived previously in the textbook, we have

$$\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(x) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)D_n(u)\ \mathrm{d}u$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{N+1}\sum_{n=0}^{N} D_n(u)\ \mathrm{d}u = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ \mathrm{d}u.$$

(b) See below for a graph of F_4, F_8, and F_{12} over the interval $[-\pi, \pi]$.



Like the Dirichlet kernel, the Fejér kernel has a large peak at 0 and decays away from 0; unlike the Dirichlet kernel, the Fejér kernel is non-negative.

Let $0 < \delta < \pi$ be given and let $A = \{u \in [-\pi, \pi] : \delta \le |u|\}$. For any $u \in A$, observe that $\sin^2\left(\frac{\delta}{2}\right) \le \sin^2\left(\frac{u}{2}\right)$. Since $\delta \in (0, \pi)$ we have $\sin^2\left(\frac{\delta}{2}\right) > 0$ and thus

$$\frac{1}{\sin^2\left(\frac{u}{2}\right)} \le \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}$$

for each $u \in A$. It follows that

$$|F_N(u)| = \frac{1}{2(N+1)} \cdot \frac{\sin^2\left((N+1)\frac{u}{2}\right)}{\sin^2\left(\frac{u}{2}\right)} \le \frac{1}{2(N+1)} \cdot \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}$$

for all $u \in A$. It is clear from this bound that $F_N \to 0$ uniformly on A.

(c) Recalling that $\int_{-\pi}^{\pi} D_n(u)\ \mathrm{d}u = \pi$ for any $n \ge 0$, we h ave

$$\int_{-\pi}^{\pi} F_N(u)\ \mathrm{d}u = \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{n=0}^{N} D_n(u)\ \mathrm{d}u = \frac{1}{N+1} \sum_{n=0}^{N} \int_{-\pi}^{\pi} D_n(u)\ \mathrm{d}u = \frac{(N+1)\pi}{N+1} = \pi.$$

(d) By assumption f is continuous on $[-\pi, \pi]$ and hence is uniformly continuous here. Thus, for any $\varepsilon > 0$, we can choose a $0 < \delta < \pi$ such that

$$|u| < \delta \quad \Rightarrow \quad |f(x+u) - f(x)| < \varepsilon.$$

For any $x \in (\pi, \pi]$ and $N \in \mathbf{N}$, parts (a) and (c) imply that

$$\sigma_N(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+u) - f(x)] F_N(u)\ \mathrm{d}u.$$

Observe that

$$\left| \frac{1}{\pi} \int_{|u|<\delta} [f(x+u) - f(x)] F_N(u)\ \mathrm{d}u \right| \le \frac{1}{\pi} \int_{|u|<\delta} [f(x+u) - f(x)] F_N(u)\ \mathrm{d}u$$

$$< \frac{\varepsilon}{\pi} \int_{|u|<\delta} F_N(u)\ \mathrm{d}u < \frac{\varepsilon}{\pi} \int_{-\pi}^{\pi} F_N(u)\ \mathrm{d}u = \varepsilon.$$

Let $M > 0$ be a bound on f over $[-\pi, \pi]$. By part (b), there exists a $K \in \mathbf{N}$ such that $F_N(u) \le \varepsilon(4M)^{-1}$ for all $\delta \le u \le \pi$ and $N \ge K$. For such N, observe that

$$\left| \frac{1}{\pi} \int_{\delta \le |u| \le \pi} [f(x+u) - f(x)] F_N(u)\ \mathrm{d}u \right| \le \frac{1}{\pi} \int_{\delta \le |u| \le \pi} |f(x+u) - f(x)| F_N(u)\ \mathrm{d}u$$

$$\le \frac{2M\varepsilon}{4M\pi} \int_{\delta \le |u| \le \pi} \mathrm{d}u < \frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}u = \varepsilon.$$

It now follows that for any $x \in (-\pi, \pi]$ and $N \ge K$ we have

$$\begin{aligned}
|\sigma_N(x) - f(x)| &= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+u) - f(x)] F_N(u) \, \mathrm{d}u \right| \\
&\le \left| \frac{1}{\pi} \int_{|u|<\delta} [f(x+u) - f(x)] F_N(u) \, \mathrm{d}u \right| \\
&\quad + \left| \frac{1}{\pi} \int_{\delta \le |u| \le \pi} [f(x+u) - f(x)] F_N(u) \, \mathrm{d}u \right| \\
&< 2\varepsilon.
\end{aligned}$$

We may conclude that $\sigma_N \to f$ uniformly on $(-\pi, \pi]$.

Exercise 8.5.11.

(a) Use the fact that the Taylor series for $\sin(x)$ and $\cos(x)$ converge uniformly on any compact set to prove WAT under the added assumption that $[a, b]$ on $[0, \pi]$.

(b) Show how the case for an arbitrary interval $[a, b]$ follows from this one.

Solution.

(a) First let us prove the following result.

Lemma L.23. Suppose that $T : \mathbf{R} \to \mathbf{R}$ is a trigonometric polynomial, i.e. T is either constant or of the form

$$T(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

for some $N \in \mathbf{N}$ and some coefficients $a_n, b_n \in \mathbf{R}$. Let $[a, b]$ be given. For any $\varepsilon > 0$, there exists a polynomial p such that $|T(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$.

Proof. If T is constant the result is clear, so suppose that T is of the form

$$T(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

for some $N \in \mathbf{N}$ and some coefficients $a_n, b_n \in \mathbf{R}$. Let $1 \le n \le N$ be given. Because the Taylor series for $\cos(nx)$ converges uniformly on $[a, b]$, there exists a polynomial p_n (some partial sum of the Taylor series) such that

$$|\cos(nx) - p_n(x)| < \frac{\varepsilon}{2N(1 + |a_n|)}$$

for each $x \in [a, b]$. Similarly, there exists a polynomial q_n such that

$$|\sin(nx) - q_n(x)| < \frac{\varepsilon}{2N(1+|b_n|)}$$

for each $x \in [a,b]$. Let p be the polynomial given by

$$p(x) = a_0 + \sum_{n=1}^{N} a_n p_n(x) + b_n q_n(x).$$

Then for any $x \in [a,b]$ we have

$$\begin{aligned}
|T(x) - p(x)| &= \left| \sum_{n=1}^{N} a_n(\cos(nx) - p_n(x)) + b_n(\sin(nx) - q_n(x)) \right| \\
&\leq \sum_{n=1}^{N} |a_n||\cos(nx) - p_n(x)| + |b_n||\sin(nx) - q_n(x)| \\
&< \sum_{n=1}^{N} \frac{\varepsilon |a_n|}{2N(1+|a_n|)} + \frac{\varepsilon |b_n|}{2N(1+|b_n|)} \\
&< \sum_{n=1}^{N} \frac{\varepsilon}{N} \\
&= \varepsilon.
\end{aligned}$$

□

Now let $f : [0,\pi] \to \mathbf{R}$ be continuous and let $\varepsilon > 0$ be given. By Fejér's Theorem (Theorem 8.5.4), $\sigma_N \to f$ uniformly on $[0,\pi]$ and thus there exists an $M \in \mathbf{N}$ such that

$$|\sigma_M(x) - f(x)| < \frac{\varepsilon}{2}$$

for all $x \in [0,\pi]$. Notice that σ_M is a trigonometric polynomial; it follows from Lemma L.23 that there exists a polynomial p such that $|\sigma_M(x) - p(x)| < \frac{\varepsilon}{2}$ for all $x \in [0,\pi]$. Thus

$$|f(x) - p(x)| \leq |\sigma_M(x) - f(x)| + |\sigma_M(x) - p(x)| < \varepsilon$$

for all $x \in [0,\pi]$.

(b) Let $f : [a,b] \to \mathbf{R}$ be continuous and define $g : [0,\pi] \to \mathbf{R}$ by $g(x) = f\left(\frac{b-a}{\pi}x + a\right)$; notice that g is continuous. Let $\varepsilon > 0$ be given. By part (a), there exists a polynomial q such that $|g(x) - q(x)| < \varepsilon$ for each $x \in [0,\pi]$. Define p by $p(x) = q\left(\frac{\pi(x-a)}{b-a}\right)$ and notice that p is a polynomial. For any $x \in [a,b]$ we have $\frac{\pi(x-a)}{b-a} \in [0,\pi]$ and thus

$$\left| g\left(\frac{\pi(x-a)}{b-a}\right) - q\left(\frac{\pi(x-a)}{b-a}\right) \right| = |f(x) - p(x)| < \varepsilon.$$

8.6. A Construction of R from Q

Exercise 8.6.1.

(a) Fix $r \in \mathbf{Q}$. Show that the set $C_r = \{t \in \mathbf{Q} : t < r\}$ is a cut.

The temptation to think of all cuts as being of this form should be avoided. Which of the following subsets of Q are cuts?

(b) $S = \{t \in \mathbf{Q} : t \leq 2\}$

(c) $T = \{t \in \mathbf{Q} : t^2 < 2 \text{ or } t < 0\}$

(d) $U = \{t \in \mathbf{Q} : t^2 \leq 2 \text{ or } t < 0\}$

Solution.

(a) It is clear that C_r satisfies (c1) and (c2). To see that C_r satisfies (c3), observe that if $t \in C_r$ then $t < \frac{t+r}{2}$ and $\frac{t+r}{2} \in C_r$.

(b) This is not a cut, since it has 2 as a maximum element.

(c) This is a cut. T satisfies (c1) since $0 \in T$ and $2 \notin T$.

Suppose $t \in T$ and r is a rational such that $r < t$. If $r < 0$ then certainly $r \in T$, so suppose that $r \geq 0$, which implies that $t > 0$. It follows that $r^2 < t^2 < 2$ and so $r \in T$. Thus T satisfies (c2).

Suppose $t \in T$. If $t \leq 0$ then let $r = 1$ and if $t > 0$ then let $r = \frac{2t+2}{t+2}$. In either case, one can verify that $t < r$ and $r \in T$. Thus T satisfies (c3).

(d) By Theorem 1.1.1 we have $U = T$ and hence U is a cut by part (c).

Exercise 8.6.2. Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Solution. Given that $r \in A$, the implication $s \notin A \Rightarrow r < s$ is the contrapositive of (c2).

Exercise 8.6.3. Using the usual definitions of addition and multiplication, determine which of these properties are possessed by $\mathbf{N}, \mathbf{Z}$, and $\mathbf{Q}$, respectively.

Solution. $\mathbf{N}$ satisfies (f1), (f2), and (f5). It fails (f3) since there is no additive identity and it fails (f4) since no element has an additive inverse and only 1 has a multiplicative inverse (1 is its own multiplicative inverse).

$\mathbf{Z}$ satisfies (f1), (f2), (f3), and (f5). It fails (f4) since, while each element has an additive inverse, only 1 and -1 have multiplicative inverses (they are their own multiplicative inverses).

$\mathbf{Q}$ satisfies each property (f1) - (f5).

Exercise 8.6.4. Show that this defines an ordering on $\mathbf{R}$ by verifying properties (o1), (o2), and (o3) from Definition 8.6.5.

Solution. Properties (o2) and (o3) are clear, so let us verify property (o1). It will suffice to show that if $B \not\subseteq A$, then $A \subseteq B$. Since B is not a subset of A, there exists some $s \in B$ such that $s \notin A$. Let $r \in A$ be given. By Exercise 8.6.2 we must have $r < s$ and so by (c2) we have $r \in B$. Thus $A \subseteq B$.

Exercise 8.6.5.

(a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.

(b) Check that addition in $\mathbf{R}$ is commutative (f1) and associative (f2).

(c) Show that property (o4) holds.

(d) Show that the cut

$$O = \{p \in \mathbf{Q} : p < 0\}$$

successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions: $A + O \subseteq A$ and $A \subseteq A + O$.)

Solution.

(a) Since A and B are non-empty, $A + B$ must also be non-empty. Since neither A nor B contains every rational number, there exist rationals $r \notin A$ and $s \notin B$. It follows from Exercise 8.6.2 that $a + b < r + s$ for every $a \in A$ and $b \in B$, so that $r + s \notin A + B$. Thus $A + B \neq \mathbf{Q}$ and we have now shown that $A + B$ satisfies (c1).

Let $a + b \in A + B$ be given. By (c3), there exist rationals $r \in A$ and $s \in B$ such that $a < r$ and $b < s$. It follows that $a + b < r + s$ and $r + s \in A + B$. Thus $A + B$ satisfies (c3).

(b) Commutativity and associativity of addition in $\mathbf{R}$ follow immediately from commutativity and associativity of addition in $\mathbf{Q}$.

(c) Let A, B, and C be cuts such that $B \subseteq C$. If $a + b \in A + B$, then $a + b \in A + C$ also since $B \subseteq C$. Thus $A + B \subseteq A + C$.

(d) Let $a + p \in A + O$ be given. Then $p < 0$, so $a + p < a$ and it follows from (c2) that $a + p \in A$ thus $A + O \subseteq A$.

Now let $a \in A$ be given. By (c3) there exists some $b \in A$ such that $a < b$. Notice that $a = b + (a - b) \in A + O$, since $a - b < 0$. It follows that $A \subseteq A + O$ and we may conclude that $A + O = A$.

Exercise 8.6.6.

(a) Prove that $-A$ defines a cut.

(b) What goes wrong if we set $-A = \{r \in \mathbf{Q} : -r \notin A\}$?

(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in Definition 8.6.4.

Solution.

(a) Since $A \neq \mathbf{Q}$, there exists a $t \notin A$. Then $-t-1 \in -A$, since $t < -(t-1) = t+1$. Thus $-A$ is non-empty. Since A is non-empty, there exists some $r \in A$. Then $-r \notin -A$, since if $t \notin A$ then $t > -(-r) = r$ by Exercise 8.6.2. Thus $-A \neq \mathbf{Q}$ and we see that $-A$ satisfies (c1).

Suppose that $r \in -A$, so that there is some $t \notin A$ such that $t < -r$, and suppose that $s < r$. Then $t < -r < -s$, demonstrating that $s \in -A$ also. Thus $-A$ satisfies (c2).

Suppose that $r \in -A$, so that there is some $t \notin A$ such that $t < -r$. Define $s = r - \frac{r+t}{2}$ and notice that $r < s$ since $0 < -r - t$. Furthermore, $s \in -A$ since

$$t \notin A \quad \text{and} \quad t < \frac{t-r}{2} = -s.$$

Thus $-A$ satisfies (c3) and we may conclude that $-A$ is a cut.

(b) This does not necessarily define a cut. For example, let C_2 be the cut $\{r \in \mathbf{Q} : r < 2\}$. Then using this definition, we find that $-C_2 = \{r \in \mathbf{Q} : r \leq -2\}$, which fails property (c3).

(c) There exists a $t \notin A$ such that $t < -r$. By Exercise 8.6.2 it must be the case that $a < t < -r$ and thus $a + r < 0$, i.e. $a + r \in O$. Thus $A + (-A) \subseteq O$.

For the reverse inclusion, let $p < 0$ be a given rational number in O. We claim that there must exist some $r \in A$ such that $r - \frac{p}{2} \notin A$, and we will prove this by contradiction. Suppose that $r - \frac{p}{2} \in A$ for all $r \in A$. Since A is a cut, there is some $r_0 \in A$. An induction argument shows that $r_0 - \frac{np}{2} \in A$ for all $n \in \mathbf{N}$. Let $q \in \mathbf{Q}$ be given and use the Archimedean property of $\mathbf{Q}$ to obtain an $n \in \mathbf{N}$ such that $r_0 - \frac{np}{2} > q$; it follows from (c2) that $q \in A$. The conclusion is that $A = \mathbf{Q}$, contradicting (c1).

Thus there is some $r \in A$ such that $r - \frac{p}{2} \notin A$. Since $r - \frac{p}{2} < r - p$, it follows that $p - r \in -A$. Then $p = r + (p - r) \in A + (-A)$, demonstrating that $O \subseteq A + (-A)$. We may conclude that $A + (-A) = O$.

Exercise 8.6.7.

(a) Show that AB is a cut and that property (o5) holds.

(b) Propose a good candidate for the multiplicative identity (1) on $\mathbf{R}$ and show that this works for all cuts $A \geq O$.

(c) Show the distributive property (f5) holds for non-negative cuts.

Solution.

(a) It is clear that AB is non-empty. If either $A = O$ or $B = O$, then it is staightforward to verify that $AB = O \neq \mathbf{Q}$. Suppose that $A > O$ and $B > O$. There exist rationals $r \notin A$ and $s \notin B$; clearly, $r, s > 0$. If $q \in AB$, then either $q < 0$ or $q = ab$ for $a \in A$, $b \in B$ and $a, b \geq 0$. By Exercise 8.6.2 we must have $a < r$ and $b < s$, so that $ab < rs$. In either case, we have $q < rs$ and thus $rs \notin AB$, demonstrating that $AB \neq \mathbf{Q}$. Thus AB satisfies (c1).

Suppose $r \in AB$ and $q < r$. If $q < 0$ then $q \in AB$, so suppose that $q \geq 0$, which implies that $r > 0$. We must then have $r = ab$ for some $a \in A, b \in B$ with $a, b > 0$. Notice that $\frac{q}{b} < a$; (c2) then implies that $\frac{q}{b} \in A$ and hence $q = \frac{q}{b} \cdot b \in AB$. Thus AB satisfies (c2).

If $A = O$ or $B = O$ then $AB = O$, which has no maximum element. Suppose that $A > O$ and $B > O$ and let $r \in AB$ be given. If $r \leq 0$ then let q be any positive rational in AB. If $r > 0$ then $r = ab$ for some $a \in A, b \in B$ with $a, b > 0$. By (c3) there exist rationals $s \in A, t \in B$ such that $a < s$ and $b < t$. Let $q = st$ and notice that $q \in AB$ and $r = ab < st = q$. In either case, there exists a $q \in AB$ with $r < q$. Thus AB satisfies (c3) and we may conclude that AB is a cut.

Property (o5) is clear from the definition of AB.

(b) Define $I = \{p \in \mathbf{Q} : p < 1\}$ and let $A \geq O$ be given. We claim that $AI = A$. Suppose that $r \in AI$. If $r < 0$ then $r \in A$, so suppose that $r \geq 0$. Thus $r = ab$ for some $a \in A$ such that $a \geq 0$ and some $0 \leq b < 1$. It follows that $ab < a$ and so by (c2) we have $r = ab \in A$. Thus $AI \subseteq A$.

Now suppose that $a \in A$. If $a \leq 0$ then (c2) implies that $2a \in A$ and thus $a = (2a) \cdot \frac{1}{2} \in AI$. If $a > 0$ then (c3) implies that there is some $r \in A$ with $a < r$. Thus $\frac{a}{r} \in I$ and we see that $a = r \cdot \frac{a}{r} \in AI$. Hence $A \subseteq AI$ and we may conclude that $AI = A$.

(c) Let $A, B, C \geq O$ be cuts. If $ABC = O$ then the equality $A(B + C) = AB + AC$ is clear, so suppose that $A, B, C > O$ and suppose that $q \in A(B + C)$. If $q < 0$ then $q = \frac{q}{2} + \frac{q}{2} \in AB + AC$. Suppose that $q \geq 0$. Then $q = a(b + c) = ab + ac$, where $a \in A, b \in B, c \in C$ and $a, b + c \geq 0$. There are three cases: $b, c \geq 0, b \geq 0$ and $c < 0$, or $b < 0$ and $c \geq 0$. In any of these cases it is straightforward to verify that $ab + ac \in AB + AC$. Thus $A(B + C) \subseteq AB + AC$.

Now suppose that $p+q \in AB+AC$. If $p+q<0$ then $p+q \in A(B+C)$, so suppose that $p+q \geq 0$. If $p, q \geq 0$ then $p=a_1 b$ and $q=a_2 c$ for some $a_1, a_2 \in A, b \in B$, and $c \in C$ such that $a_1, a_2, b, c \geq 0$. Let $a=\max\{a_1, a_2\}$ and notice that $a(b+c) \in A(B+C)$. Furthermore, $p+q=a_1 b+a_2 c \leq ab+ac=a(b+c)$. It follows from (c2) that $p+q \in A(B+C)$.

Next, suppose that $p<0$ and $q \geq 0$, so that $q=ac$ for some $a \in A, c \in C$ with $a, c \geq 0$. Let $b \in B$ be such that $b \geq 0$; such a b exists since $B>O$. Now notice that

$$p+q=p+ac<ac \leq a(b+c) \in A(B+C).$$

It follows from (c2) that $p+q \in A(B+C)$. The case where $p \geq 0$ and $q<0$ is handled similarly. Thus $AB+AC \subseteq A(B+C)$ and we may conclude that $A(B+C)=AB+AC$.

Exercise 8.6.8. Let $\mathcal{A} \subseteq \mathbf{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.

(a) First, prove that $S \in \mathbf{R}$ by showing that it is a cut.

(b) Now, show that S is the least upper bound for $\mathcal{A}$.

Solution.

(a) Since $\mathcal{A}$ is non-empty it contains some cut A, so that $A \subseteq S$. It follows that S is non-empty as A is non-empty. Since $\mathcal{A}$ is bounded above, there exists some cut B such that $A \subseteq B$ for all $A \in \mathcal{A}$. It follows that $S \subseteq B$ and hence that $S \neq \mathbf{Q}$ since $B \neq \mathbf{Q}$. Thus S satisfies (c1).

Suppose $r \in S$, so that $r \in A$ for some $A \in \mathcal{A}$, and suppose $q<r$. Since A is a cut we must have $q \in A$, which gives $q \in S$. Thus S satisfies (c2).

Suppose $r \in S$, so that $r \in A$ for some $A \in \mathcal{A}$. Since A is a cut there exists some $q \in A$ such that $r<q$; note that $q \in S$ also. Thus S satisfies (c3). We may conclude that S is a cut.

(b) It is clear that S is an upper bound for $\mathcal{A}$. If B is any upper bound for $\mathcal{A}$ then B contains every $A \in \mathcal{A}$ and hence must contain the union of all $A \in \mathcal{A}$, i.e. $S \subseteq B$. We may conclude that S is a cut.

Exercise 8.6.9. Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbf{Q} : t < r\}$$

where $r \in \mathbf{Q}$. (See Exercise 8.6.1.)

(a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbf{Q}$., Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.

(b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in $\mathbf{Q}$.

Solution.

(a) Let $r, s \in \mathbf{Q}$ be given and suppose $a + b \in C_r + C_s$, i.e. $a < r$ and $b < s$. It follows that $a + b < r + s$ and hence that $a + b \in C_{r+s}$. Thus $C_r + C_s \subseteq C_{r+s}$. Now suppose that $t \in C_{r+s}$, so that $t < r + s$. Choose a positive integer $n \in \mathbf{N}$ such that $t + \frac{1}{n} < r + s$ and note that:

- $s - \frac{1}{n} < s$, so that $s - \frac{1}{n} \in C_s$;
- $t + \frac{1}{n} - s < r$, so that $t + \frac{1}{n} - s \in C_r$;
- $t = \left(t + \frac{1}{n} - s\right) + \left(s - \frac{1}{n}\right) \in C_r + C_s$.

Thus $C_{r+s} \subseteq C_r + C_s$ and we may conclude that $C_r + C_s = C_{r+s}$.

It is clear that $C_r C_s = C_{rs}$ if $rs = 0$, so suppose that $r, s > 0$ and let $q \in C_r C_s$ be given. If $q \leq 0$ then $q < rs$, i.e. $q \in C_{rs}$. If $q > 0$ then $q = ab$ for some $0 < a < r$ and $0 < b < c$. It follows that $0 < ab < rs$ and thus $q = ab \in C_{rs}$. Hence $C_r C_s \subseteq C_{rs}$.

Now let $q \in C_{rs}$ be given. If $q \leq 0$ then certainly $q \in C_r C_s$ so suppose that $q > 0$ and define $p = \frac{1}{2}\left(\frac{q}{s} + r\right)$. Notice that:

- $0 < \frac{q}{s} < p < r$, so that $p \in C_r$;
- $0 < \frac{q}{p} < s$, so that $\frac{q}{p} \in C_s$;
- $q = p \cdot \frac{q}{p} \in C_r C_s$.

Thus $C_{rs} \subseteq C_r C_s$ and we may conclude that $C_r C_s = C_{rs}$.

(b) If $r \leq s$ then it is clear that $C_r \subseteq C_s$. If $s < r$ then it is again clear that $C_s \subseteq C_r$. Furthermore, notice that $C_s \neq C_r$ since $\frac{s+r}{2}$ belongs to C_r but not to C_s. Thus $C_s \subsetneq C_r$.